Lecture X: Hochster's Formula and an Example (March 14, 2006)
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We have two goals in this lecture:

(1) State Hochster's Formula; this formula relates the reduced simplicial homology of Δ with the graded Betti numbers of the Stanley-Reisner ideal I_Δ.
(2) Use an example to explore this formula and other previous results.

1. Hochster's Formula

Suppose Δ is a simplicial complex on $V = \{x_1, x_2, \ldots, x_n\}$. For any $W \subseteq V$, the restriction of Δ to W is the simplicial complex $\Delta_W = \{F \in \Delta \mid F \subseteq W\}$.

Theorem 1.1 (Hochster's Formula)**.** *Let Δ be a simplicial complex on $V = \{x_1, \ldots, x_n\}$ and let I_Δ be the associated Stanley-Reisner ideal in $R = k[x_1, x_2, \ldots, x_n]$. Then*

$$\beta_{i,j}(I_\Delta) = \sum_{|W|=j, W \subseteq V} \dim_k \widetilde{H}_{j-i-2}(\Delta_W, k)$$

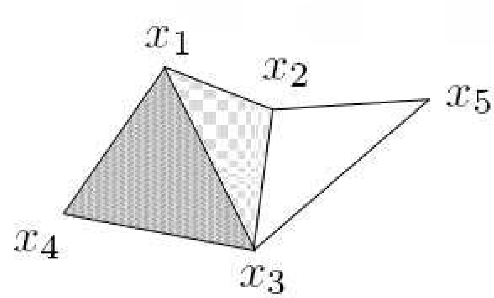
where by $\widetilde{H}_l(\Gamma, k)$ we mean the reduced simplicial homology of Γ.

Corollary 1.2. *If $j > |V| = n$, then $\beta_{i,j}(I_\Delta) = 0$.*

Proof. Since there is no $W \subseteq V$ with $|W| = j > |V| = n$, the conclusion follows from Hochster's formula. □

2. Worked out example

Hochster's formula is a beautiful result but somewhat daunting to use. We will examine this formula (and other results introduced in these lectures) in relation to the simplicial complex Δ:



Note that for this lecture, the tetrahedron formed by the vertices x_1, x_2, x_3 and x_4 is hollow.

2.1. **Dimension, facets, and f-vectors.** The facets of Δ are:

$$\{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}\}.$$

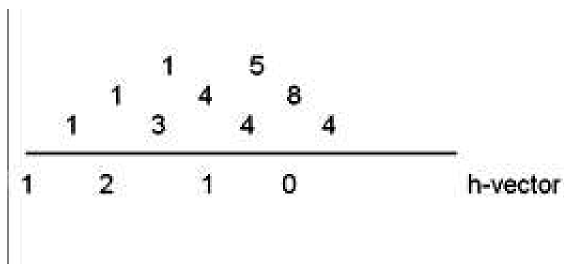
Thus, we have $\dim \Delta = \max\{\dim F \mid F \in \Delta\} = 2 = |\{x_1, x_2, x_3\}| - 1$. The f-vector of Δ has the form $f(\Delta) = (f_0, f_1, f_2)$. For this simplicial complex $f(\Delta) = (5, 8, 4)$ because there are 5 vertices, 8 edges, and 4 triangles.

2.2. **Stanley-Reisner ideal.** Since $V = \{x_1, \dots, x_5\}, R = k[x_1, \dots, x_5]$. The Stanley-Reisner ideal is given by

$$I_\Delta = (x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \dots, x_{i_r}\} \notin \Delta) = (\underbrace{x_1x_5, x_4x_5}_{\textit{missing edges}}, \underbrace{x_2x_3x_5}_{\textit{missing triangle}}, \underbrace{x_1x_2x_3x_4}_{\textit{missing tetrahedron}}).$$

Note that I_Δ is generated by the minimal non-faces. Furthermore, since $\dim \Delta = 2$, the Krull dimension of R/I_Δ equals $\dim R/I_\Delta = \dim \Delta + 1 = 3$.

2.3. **Hilbert-Series, h-vector, and Hilbert function.** Since the f-vector $f(\Delta) = (5, 8, 4)$, we can find the h-vector using a "Pascal-triangle like" method. Using $(5, 8, 4)$ as one edge of the triangle, we get



The h-vector is $h(\Delta) = (1, 2, 1, 0)$. Since $\dim R/I_\Delta = 3$, the Hilbert series of R/I_Δ is given by

$$HS(R/I_\Delta, t) = \frac{1 + 2t + t^2}{(1 - t)^3} = (1 + 2t + t^2)(1 + 3t + 6t^2 + 10t^3 + \cdots)$$

By expanding out this infinite series, the coefficient of t^d is value of the Hilbert function at d, i.e.

$$H_{R/I_\Delta}(d) = \dim_k(R/I_\Delta)_d = \text{ coefficient of } t^d.$$

2.4. **Resolution.** Since I_Δ is a homogeneous ideal of $R = k[x_1, x_2, x_3, x_4, x_5]$, the ideal has a minimal free graded resolution of the form

$$0 \longrightarrow \mathcal{F}_l \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow I_\Delta \longrightarrow 0$$

with $l \leq 5 = n$ and $\mathcal{F}_i = R(-d_{i,1}) \oplus \ldots \oplus R(-d_{i,t_i})$. Recall that $\beta_{i,j}(I_\Delta)$ is the number of times $R(-j)$ appears in $\mathcal{F}_i$. We will use Hochster's formula to compute the resolution.

Step 0. Find $\mathcal{F}_0$.

For the 0^{th} graded Betti numbers, i.e. $\beta_{0,j}(I_\Delta)$, we only need to know the generators of I_Δ. This is because $\beta_{0,j}(I_\Delta)$ is the number of generators of degree j. Since

$$I_\Delta = (x_1x_5, x_4x_5, x_2x_3x_5, x_1x_2x_3x_4)$$

we have $\beta_{0,2}(I_\Delta) = 2, \beta_{0,3}(I_\Delta) = 1, \beta_{0,4}(I_\Delta) = 1$, and $\beta_{0,j}(I_\Delta) = 0$ otherwise. Thus

$$\mathcal{F}_0 = R^2(-2) \oplus R(-3) \oplus R(-4).$$

Step 1. Find $\mathcal{F}_1$.

We need to compute $\beta_{1,j}(I_\Delta)$ for all $j \in \mathbb{N}$. By Hochster's Formula

$$\beta_{1,j}(I_\Delta) = \sum_{W \subseteq V, |W|=j} \dim_k \widetilde{H}_{j-1-2}(\Delta_W, k) = \sum_{W \subseteq V, |W|=j} \dim_k \widetilde{H}_{j-3}(\Delta, k)$$

If $j > 5$, then $\beta_{i,j}(I_\Delta) = 0$ since there is no $W \subseteq V$ with $|W| = j > 5$. On the other hand, if $j \leq 1$, then $j - 3 < -1$. But for any simplicial complex Γ, $\widetilde{H}_l(\Gamma, k) = 0$ if $l < -1$. So $\beta_{1,j}(I_\Delta) = 0$ if $j \leq 1$. Thus, we only need to compute $\beta_{1,j}(I_\Delta)$ for $j = 2, 3, 4, 5$. We consider each j separately.

j =2

We will require the following lemma.

Lemma 2.1. *If $\Gamma \neq \emptyset$, then $\widetilde{H}_{-1}(\Gamma, k) = 0$.*

Suppose $W \subseteq V$ is a subset with $|W| = 2$. Then $\Delta_W \neq \{\emptyset\}$ since Δ_W contains at least the two vertices of W. By Lemma 2.1

$$\dim_k \widetilde{H}_{2-3}(\Delta_W, k) = \dim_k \widetilde{H}_{-1}(\Delta_W, k) = 0.$$

So $\beta_{1,2}(I_\Delta) = 0$.

j =3

We require another lemma to compute $\beta_{1,3}(I_\Delta)$.

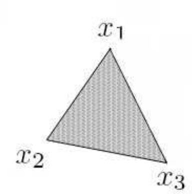
Lemma 2.2. *For any simplicial complex Γ,*

$$\dim_k \widetilde{H}_0(\Gamma, k) + 1 = \text{the number of connected components of } \Gamma.$$

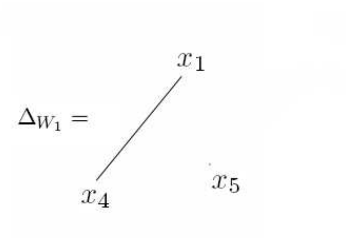
Let $W \subseteq V$ with $|W| = 3$. So

$$\dim_k \widetilde{H}_{3-3}(\Delta_W, k) = (\#\text{ of connected components of } \Delta_W) - 1.$$

For example, if $W = \{x_1, x_2, x_3\}$, then $\Delta_W =$



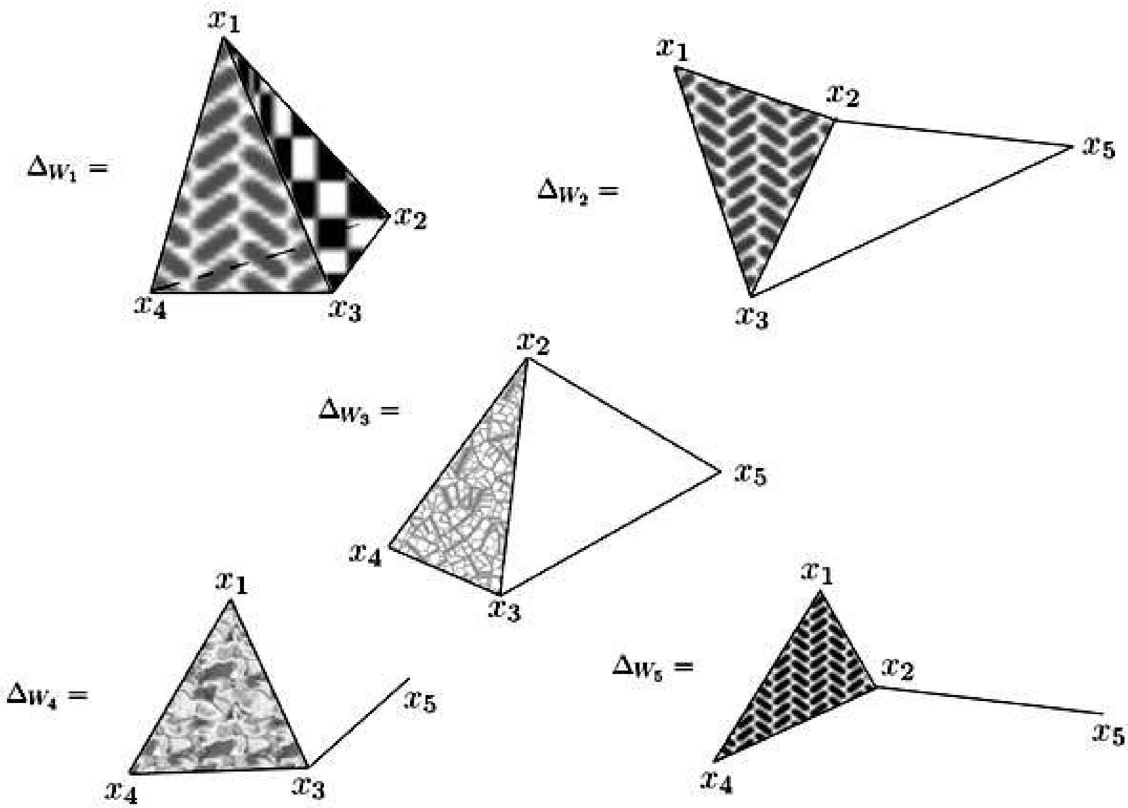
so $\dim_k \widetilde{H}_0(\Delta_W, k) = 0$. If $W = \{x_1, x_4, x_5\}$, then $\Delta_W =$



Then $\dim_k \widetilde{H}(\Delta_W, k) = 2 - 1 = 1$. By checking all $\binom{5}{3} = 10$ subsets $W \subseteq V$ with $|W| = 3$, the only subset W with Δ_W not connected is $W = \{x_1, x_4, x_5\}$. So $\beta_{1,3}(I_\Delta) = \dim_k \widetilde{H}_0(\Delta_W, k) = 1$.

j =4

There are only 5 subsets of V containing 4 vertices. Each subset gives a different restriction of Δ. The five restrictions are



We need to compute $\dim_k \widetilde{H}_{4-3}(\Delta_{W_i}, k) = \dim_k \widetilde{H}_1(\Delta_{W_i}, K)$ for $i = 1, \ldots, 5$. Note that $\Delta_{W_2} = \Delta_{W_3}$ and $\Delta_{W_4} = \Delta_{W_5}$ so we only need to compute the dimension of the homology groups of three simplicial complexes. We omit the calculations here, but provided the dimensions of the following homology groups:

Claim:

$$\begin{aligned} \dim_k \widetilde{H}_1(\Delta_{W_1}, k) &= 0 \\ \dim_k \widetilde{H}_1(\Delta_{W_2}, k) = \dim_k \widetilde{H}_1(\Delta_{W_3}, k) &= 1 \\ \dim_k \widetilde{H}_1(\Delta_{W_4}, k) = \dim_k \widetilde{H}_1(\Delta_{W_5}, k) &= 0. \end{aligned}$$

Note that the nonzero values come from fact that Δ_{W_2} and Δ_{W_3} have "holes", i.e., the hole formed by $\{x_2, x_3, x_5\}$ in both simplicial complexes. So $\beta_{1,4}(I_\Delta) = 2$ (each of Δ_{W_2} and Δ_{W_3} contribute 1 to $\beta_{1,4}(I_\Delta)$).

j =5

In this case, $\Delta_W = \Delta$. We compute

$$\dim_k \widetilde{H}_{5-3}(\Delta_W, k) = \dim_k \widetilde{H}_2(\Delta_W, k).$$

Now

$$\widetilde{H}_2(\Delta, k) = \frac{\ker \partial_2}{\mathrm{Im}\partial_3}$$

where the maps ∂_2 and ∂_3 come from

$$0 \xrightarrow{\partial_3} k^{F_2(\Delta)} \xrightarrow{\partial_2} k^{F_1(\Delta)} \longrightarrow \cdots$$

Recall that

$$F_2(\Delta) = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\}$$
$$F_1(\Delta) = \{\{x_1, x_2\}, \ldots, \{x_3, x_5\}\} \text{ (the eight edges)}$$

and $k^{F_i(\Delta)}$ is the k-vector space whose bases elements are the elements of $F_i(\Delta)$.

Now $\mathrm{Im}\partial_3 = 0$, so $\widetilde{H}_2(\Delta, k) = \ker \partial_2$. Note that

$$k^{F_2(\Delta)} = \{c_1\{x_1, x_2, x_3\} + c_2\{x_1, x_2, x_4\} + c_3\{x_1, x_3, x_4\} + c_4\{x_2, x_3, x_4\} \mid c_i \in k\} \cong k^4.$$

We will show that $\ker \partial_2$ is a one dimensional vector space.

$$\begin{aligned} &\partial_2(c_1\{x_1, x_2, x_3\} + c_2\{x_1, x_2, x_4\} + c_3\{x_1, x_3, x_4\} + c_4\{x_2, x_3, x_4\}) \\ &= c_1\{x_2, x_3\} - c_1\{x_1, x_3\} + c_1\{x_1, x_2\} + c_2\{x_2, x_4\} - c_2\{x_1, x_4\} + c_2\{x_1, x_2\} + \\ &c_3\{x_3, x_4\} - c_3\{x_1, x_4\} + c_3\{x_1, x_3\} + c_4\{x_3, x_4\} - c_4\{x_2, x_4\} + c_4\{x_2, x_3\} = 0 \end{aligned}$$

We see that a 3-chain is sent to zero if and only if $c_1 = -c_2 = -c_4 = c_3$. So

$$\ker \partial_2 = \{c(\{x_1, x_2, x_3\} - \{x_1, x_2, x_4\} + \{x_1, x_3, x_2\} - \{x_2, x_3, x_4\}) \mid c \in k\} \cong k.$$

Thus $\dim_k \widetilde{H}_2(\Delta, k) = 1$, which, in turn, implies that $\beta_{1,5}(I_\Delta) = 1$. The non-zero homology is coming from the fact that Δ contains the boundary of the tetrahedron $\{x_1, x_2, x_3, x_4\}$, but not the face $\{x_1, x_2, x_3, x_4\}$. Thus, for Step 1, we have shown that

$$\mathcal{F}_1 = R(-3) \oplus R^2(-4) \oplus R(-5).$$

Step 2. Find $\mathcal{F}_2$.

By arguments similar to those used in Step 1, $\beta_{2,j}(I_\Delta) = 0$ if $j > 5$ or $j \leq 3$. So we need to find $\beta_{2,j}(I_\Delta)$ where $j = 4, 5$. We consider each case separately.

j =4

For each $W \subseteq V$, with $|W| = 4$, we need to compute $\dim_k \widetilde{H}_{4-2-2}(\Delta_W, k) = \dim_k \widetilde{H}_0(\Delta_W, k)$ $=$ (# connected components of Δ_W) -1. But for every such W, Δ_W is connected. So $\dim_k \widetilde{H}_0(\Delta_W, k) = 0$. Thus $\beta_{2,4}(I_\Delta) = 0$.

j =5

Since $W = V, \Delta_W = \Delta$. We claim (but don't verify) that $\dim_k \widetilde{H}_1(\Delta, k) = 1$. The one is coming from the fact that Δ has a "hole" formed by the vertices $\{x_2, x_3, x_5\}$. Thus $\beta_{2,5}(I_\Delta) = 1$ which implies that

$$\mathcal{F}_2 = R(-5).$$

Step 3. Show for $i \leq 3, \beta_{i,j}(I_\Delta) = 0$

This step has been left as an exercise.

We summarize the results on the resolution in the following corollary:

Corollary 2.3. *The minimal free graded resolution of I_Δ has the form*

$$0 \longrightarrow R(-5) \longrightarrow R(-3) \oplus R^2(-4) \oplus R(-5) \longrightarrow R^2(-2) \oplus R(-4) \oplus R(-3) \longrightarrow I_\Delta \longrightarrow 0.$$

Problems from Lecture 9

1. Let Δ be the simplicial complex studied in the lecture. Show that if $i \geq 3$, then $\beta_{i,j}(I_\Delta) = 0$. **Hint:** Consider the cases that $i \geq 4$ and $i = 3$ separately.
2. Let Δ be the simplicial complex studied in the lecture. Show that if $W = \{x_1, x_3, x_4, x_5\}$, then $\widetilde{H}_1(\Delta_W, k) = 0$.
3. Show that if $\Gamma \neq \{\emptyset\}$, then $\widetilde{H}_{-1}(\Gamma, k) = 0$.