NAME: _____

STUDENT NUMBER: _____

MATH 1281 - Final Exam

Lakehead University

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Instructions: Answer all questions in the space provided. If you need more room, answer on the back of the page. Where appropriate, you must provide clear explanations.

You are *not* allowed to use a calculator. If a question involves a calculation you may leave it in an unexpanded form, e.g., you can write 5^4 instead of 625.

If doubt exists as to the interpretation of any question, you are urged to submit with the answer paper a clear statement of any assumptions made.

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Chapter 1 – Logics and Proofs

1. **[2pts]** Construct a truth table for the following compound proposition:

$$(\neg p \vee q) \rightarrow r$$

2. **[4pts]** Let the universe of discourse be $D = \{2, 4, 6, 8, 10, 12\}$, and consider the following propositional functions:

$$Q(x) = x \text{ is even}, \qquad S(x) = x^2 < 1, \qquad T(x) = x - 2 \text{ is an element of } D.$$

Circle all the **true** statements in the list below:

$$\forall x Q(x) \quad \exists x Q(x) \quad \forall x \neg Q(x) \quad \exists x \neg Q(x)$$

$$\forall x S(x) \quad \exists x S(x) \quad \forall x \neg S(x) \quad \exists x \neg S(x)$$

$$\forall x T(x) \quad \exists x T(x) \quad \forall x \neg T(x) \quad \exists x \neg T(x)$$

3. **[4pts]** Prove that the product of two odd numbers is odd.

Chapter 2 – Sets, Functions, Sequences, and Sums

4. **[2pts]** Provide a simple formula or rule that generates the terms of the integer sequence that begins:

$$2, 5, 8, 11, 14, 17, \ldots$$

5. **[2pts]** Determine if the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) \quad = \quad \left\lfloor \frac{x^2}{2} \right\rfloor$$

is one-to-one (injective) and/or onto (surjective). Justify your answer.

6. **[4pts]** Let A, B and C be sets. Is it true that

$$(A - B) - C = A - (B - C)?$$

If yes, prove the set equality; if no, give a counterexample.

Chapter 3 – Algorithms, Integers and Matrices

7. [**4pts**] Consider the following algorithm:

```
procedure mystery(a_1,...,a_n; integers)

max:=a_1
min:=a_1
for i:=1 to n
begin
   if a_i > max then max:=a_i
   if a_i < min then min:=a_i
end
mystery:=(max+min)/2
return(mystery)
end
```

What is the output of this algorithm with the input `mystery(1,2,3,4,5,6,7,8,9,10)`?

8. [**4pts**] Let m and n be positive integers such that $n|m$ and $m, n > 1$. Prove that if $a \equiv b \pmod{m}$, then $a \equiv b \pmod{n}$.

Chapter 4 – Induction and Recursion

9. [**4pts**] Prove by mathematical induction that $n^2 \geq 2n + 1$ when $n \geq 3$.

10. **[4pts]** Prove by mathematical induction that for all natural numbers $n \geq 1$,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Chapter 5 – Counting

11. **[2pts]** What is the coefficient of $x^{16}y^4$ in the expansion of $(3x-2y)^{20}$?

12. **[4pts]** Count the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 34$$

where x_1, x_2, x_3 and x_4 are integers and when

(*i*) $x_1, x_2, x_3, x_4 \geq 0$.

(*ii*) $x_1, x_2, x_3 \geq 0$, and $0 \leq x_4 \leq 6$.

You may leave your solution in an unexpanded form using binomial coefficients.

Chapter 6 – Discrete Probability

13. [**2pts**] Suppose you have two fair 8 sided die. What is probability of rolling a 10?

14. [**4pts**] Find and correct the error in the solution to the following problem:

Problem. What is the probability that all heads appear when you flip three coins?

Solution. There are four possible outcomes: (i) all heads, (ii) two heads and one tail, (iii) one head and two tails, and (iv) three tails. So, $p(\text{all heads}) = 1/4$.

Chapter 7 – Advance Counting Techniques

15. [**4pts**] If $G(x)$ is the generating function for $a_0, a_1, a_2, \ldots$, describe in terms of $G(x)$ the generating function for

(*i*) $0, 0, 0, a_0, a_1, a_2, \ldots$

(*ii*) $a_0, 3a_1, 9a_2, 27a_3, 81a_4 \ldots$

16. **[4pts]** Solve the recurrence relation

$$9a_n = 6a_{n-1} - a_{n-2}$$

when $a_0 = 6$ and $a_1 = 5$.

Chapter 8 – Relations

17. **[6pts]** Consider the set $A = \{-2, -1, 0, 1, 2\}$ and consider the relation R on A where

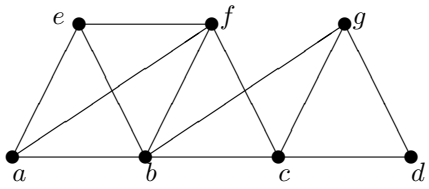
$$R = \{(a, b) \mid a^2 = b^2\}.$$

(*i*) What is the zero-one matrix associated to R?

(*ii*) Determine if R is (a) reflexive, (b) symmetric, (c) antisymmetric, and (d) transitive. Justify your answers!

Chapter 9 – Graphs

18. For the questions below, use the following graph:



(*i*) **[2pts]** Write out the adjacency matrix for the above graph.

(*ii*) **[2pts]** Find a subgraph of the above graph isomorphic to C_4.

(*iii*) **[2pts]** Is the above graph a planar graph? If yes, draw the graph as a planar graph; if no, explain why not.

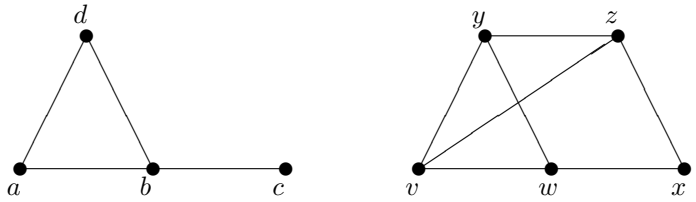
(*iv*) **[2pts]** Does the above graph have an Euler path? an Euler circuit? If so, write out the path/circuit.

(*v*) **[2pts]** Does the above graph have a Hamilton path? a Hamilton circuit? If so, write out the path/circuit.

(*vi*) **[2pts]** Find the chromatic number of the above graph.

19. **[6pts]** We say that connected graph is **almost Eulerian** if it contains a circuit that uses every edge in the graph once and one edge in the graph twice.

(*i*) Determine if the following two graphs are almost Eulerian:



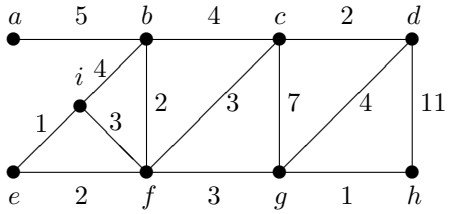
(*ii*) Find a characterization of *almost Eulerian* graphs similar to the characterization of Euler graphs given in class.

20. **[2pts]** Draw the graph whose adjacency matrix has the form

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chapter 10 – Trees

21. Answer the following questions about the weighted graph given below:



(*i*) **[2pts]** Find the shortest path between a and h (you can do this by inspection).

(*ii*) **[3pts]** Use Prim's algorithm to find a minimal spanning tree of the above graph. As in class, list the order in which you picked the edges for your spanning tree, and draw your spanning tree.

(*iii*) **[3pts]** Now adapt Kruskal's Algorithm to find a **maximal spanning tree**, i.e., a spanning tree of largest weight. As in class, list the order in which you picked the edges for your spanning tree, and draw your spanning tree.

22. **[2pts]** Create a binary search tree for the following list of words in the quote:

Black holes result from God dividing the universe by zero.

(Assume that normal alphabetical ordering is being used.)

Chapter 11 – Boolean Algebras

23. **[2pts]** Using a table, express the values of the Boolean function:

$$F(x, y, z) = x(yz + \overline{x}\ \overline{z}).$$

24. In the strange world of ANTI-LAND, democracy works backwards, that is, the minority makes the decision. So, in a committee of three people, a motion will pass if 1 or less people vote for it, but the motion will fail if 2 or more people vote for it.

(*i*) **[4pts]** Design a circuit that implements the voting of this committee using inverters, AND gates, and OR gates.

(*ii*) **[4pts]** Can you simplify your circuit in (*i*)? If so, do so; otherwise, explain why you cannot simplify your circuit.