NAME: Solutions

STUDENT NUMBER: ____________

MATH 1281 - Final Exam

Lakehead University

April 22, 2008

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Instructions: Answer all questions in the space provided. If you need more room, answer on the back of the page. Where appropriate, you must provide clear explanations.

You are *not* allowed to use a calculator. If a question involves a calculation you may leave it in an unexpanded form, e.g., you can write 5^4 instead of 625.

If doubt exists as to the interpretation of any question, you are urged to submit with the answer paper a clear statement of any assumptions made.

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Chapter 1 – Logics and Proofs

1. **[2pts]** Construct a truth table for the following compound proposition:

$$(\neg p \vee q) \rightarrow r$$

P	Q	R	¬P	¬P ∨ Q	(¬P ∨ Q) → R
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F

2. **[4pts]** Let the universe of discourse be $D = \{2, 4, 6, 8, 10, 12\}$, and consider the following propositional functions:

✗ $\quad Q(x) = x$ is even. $\quad S(x) = x^2 < 1.$ $\quad T(x) = x - 2$ is an element of D.

Circle all the **true** statements in the list below:

$\forall x Q(x)$ $\quad \exists x Q(x)$ $\quad \forall x \neg Q(x)$ $\quad \exists x \neg Q(x)$

$\forall x S(x)$ $\quad \exists x S(x)$ $\quad \forall x \neg S(x)$ $\quad \exists x \neg S(x)$

$\forall x T(x)$ $\quad \exists x T(x)$ $\quad \forall x \neg T(x)$ $\quad \exists x \neg T(x)$

3. **[4pts]** Prove that the product of two odd numbers is odd.

Let x and y be odd numbers. So $x = 2m+1$ and $y = 2n+1$ for some integers m and n. Then

$$xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$$
$$= 2(2mn + m + n) + 1$$

Since xy has form $2\ell + 1$, xy is odd. $\boxed{\checkmark}$

Chapter 2 – Sets, Functions, Sequences, and Sums

4. **[2pts]** Provide a simple formula or rule that generates the terms of the integer sequence that begins:

$$2, 5, 8, 11, 14, 17, \ldots$$

Arithmetic sequence with $a_0 = 2$ and $d = 3$.
So $a_n = 2 + 3 \cdot n$ for $n \geq 0$

Alt. answer
$a_0 = 2$
$a_n = a_{n-1} + 3$ for $n \geq 1$

5. **[2pts]** Determine if the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \left\lfloor \frac{x^2}{2} \right\rfloor$$

* is one-to-one (injective) and/or onto (surjective). Justify your answer.

f is not one-to-one since $f(0) = \left\lfloor \frac{0^2}{2} \right\rfloor = 0 = \left\lfloor \frac{1^2}{2} \right\rfloor = f(1)$

f is not surjective since ~~for any~~ there is no integer $n \in \mathbb{N}$ such that $f(n) = \left\lfloor \frac{n^2}{2} \right\rfloor = 3$. To see this, note that

$f(2) = \left\lfloor \frac{2^2}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2$ and $f(3) = \left\lfloor \frac{9}{2} \right\rfloor = 4$. Since $f(x)$ is always increasing, no integer will give us a value of three.

6. **[4pts]** Let A, B and C be sets. Is it true that

$$(A - B) - C = A - (B - C)?$$

If yes, prove the set equality; if no, give a counterexample.

False: Let $A = \{1,2,3\}$
$B = \{1\}$
$C = \{1,4\}$

Then $(A-B) - C = \{2,3\} - \{1,4\} = \{2,3\}$ not equal
and $A - (B - C) = A - \emptyset = A = \{1,2,3\}$

Chapter 3 – Algorithms, Integers and Matrices

7. **[4pts]** Consider the following algorithm:

```
procedure mystery(a_1,...,a_n; integers)

max:=a_1
min:=a_1
for i:=1 to n
begin
   if a_i > max then max:=a_i
   if a_i < min then min:=a_i
end
mystery:=(max+min)/2
return(mystery)
end
```

What is the output of this algorithm with the input `mystery(1,2,3,4,5,6,7,8,9,10)`?

We will have max=10 min=1, so mystery will return

$$\boxed{\frac{10+1}{2} = \frac{11}{2}}$$

8. **[4pts]** Let m and n be positive integers such that $n|m$ and $m, n > 1$. Prove that if $a \equiv b \pmod{m}$, then $a \equiv b \pmod{n}$.

Suppose that $a \equiv b \bmod m$. So $m | a-b$, i.e.

$(a-b) = md$ for some integer d.

Now we know that $n|m$, so $m = ne$ for some integer e.

Hence. $(a-b) = (ne)d = n(ed)$.

Thus, $n | (a-b)$ with $n > 1$. But this just means

$a \equiv b \bmod n$.

$\boxed{5}$

Chapter 4 – Induction and Recursion

9. [**4pts**] Prove by mathematical induction that $n^2 \geq 2n+1$ when $n \geq 3$.

Let $P(n) =$ " $n^2 \geq 2n+1$ "

Base Case: Our base case is $n=3$. $P(3)$ is true since

$P(3) =$ " $3^2 = 9 \geq 7 = 2 \cdot 3 + 1$ ". So, our base case is true.

Induction step: Assume $P(n)$ is true, i.e. $n^2 \geq 2n+1$.

We now want to show that $P(n+1)$ is true, i.e.

$$(n+1)^2 \geq 2(n+1)+1 = 2n+3$$

So, we have

$$(n+1)^2 = n^2 + 2n + 1$$
$$\geq (2n+1) + (2n+1) \quad \text{by induction since } n^2 \geq 2n+1$$

Because $n \geq 3$, $2n+1 \geq 2 \cdot 3 + 1 = 7 \geq 2$.

Thus $(n+1)^2 \geq (2n+1) + (2n+1) \geq (2n+1) + 2 = 2n+3$.

So $(n+1)^2 \geq 2n+3 = 2(n+1)+1$, and thus, by induction, $P(n)$ is true for all $n \geq 3$. $\boxed{?}$

10. [**4pts**] Prove by mathematical induction that for all natural numbers $n \geq 1$,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Let $P(n) =$ " $1 \cdot 3 + 2 \cdot 4 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$ "

Base Case $P(1)$ is true since

$P(1) =$ " $1 \cdot 3 = 3 = \frac{1(1+1)(2 \cdot 1 + 7)}{6} = \frac{18}{6}$ "

So, the base case is true.

Induction Step Suppose $P(n)$ is true, i.e.

$$1 \cdot 3 + \ldots + (n)(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

want to show $P(n+1)$ is true, i.e. want to show

$$1 \cdot 3 + \ldots + n(n+2) + (n+1)(n+3) = \frac{(n+1)(n+2)(2n+9)}{6}$$

Now

$$1 \cdot 3 + \ldots + n(n+2) + (n+1)(n+3) = \frac{n(n+1)(2n+7)}{6} + (n+1)(n+3)$$ by induction hypothesis

$$= \frac{n(n+1)(2n+7) + 6(n+1)(n+3)}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6n + 18)}{6}$$

$$= \frac{(n+1)(n+2)(2n+9)}{6}$$

Thus, $P(n+1)$ is true if $P(n)$ is true.

$\therefore$ by mathematical induction, $P(n)$ is true for all $n \geq 1$. $\boxtimes$

Chapter 5 – Counting

11. **[2pts]** What is the coefficient of $x^{16}y^4$ in the expansion of $(3x-2y)^{20}$?

use binomial theorem: $\binom{20}{16} 3^{16} (-2)^4$

12. **[4pts]** Count the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 34$$

where x_1, x_2, x_3 and x_4 are integers and when

(*i*) $x_1, x_2, x_3, x_4 \geq 0$.

(*ii*) $x_1, x_2, x_3 \geq 0$, and $0 \leq x_4 \leq 6$.

You may leave your solution in an unexpanded form using binomial coefficients.

(i) $\binom{34+4-1}{34} = \binom{37}{34}$

(ii) # of sol'ns w/ $x_1, x_2, x_3 \geq 0$ and $0 \leq x_4 \leq 6$

= # sol'ns w/ $x_1, x_2, x_3, x_4 \geq 0$

− # sol'ns w/ $x_1, x_2, x_3 \geq 0$ and $x_4 \geq 7$

$= \binom{37}{34} - \binom{27+4-1}{27}$

$= \binom{37}{34} - \binom{30}{27}$

Chapter 6 – Discrete Probability

13. **[2pts]** Suppose you have two fair 8 sided die. What is probability of rolling a 10?

There are $64 = 8^2$ possible outcomes.

There are 7 ways to roll a 10: $(2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2)$

$p(\text{rolling a ten}) = \frac{7}{64}$

14. **[4pts]** Find and correct the error in the solution to the following problem:

Problem. What is the probability that all heads appear when you flip three coins?

Solution. There are four possible outcomes: (i) all heads, (ii) two heads and one tail, (iii) one head and two tails, and (iv) three tails. So, $p(\text{all heads}) = 1/4$.

The # of outcomes is counted incorrectly. Possible outcomes:

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

So $p(\text{all heads}) = 1/8$

Chapter 7 – Advance Counting Techniques

15. **[4pts]** If $G(x)$ is the generating function for $a_0, a_1, a_2, \ldots$, describe in terms of $G(x)$ the generating function for

x (i) $0, 0, 0, a_0, a_1, a_2, \ldots$

x (ii) $a_0, 3a_1, 9a_2, 27a_3, 81a_4, \ldots$ (i)

$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

(i) $H(x) = 0 + 0 \cdot x + 0 \cdot x^2 + a_0 x^3 + a_1 x^4 + a_2 x^5 + \ldots$

$= x^3 (a_0 + a_1 x + a_2 x^2 + \ldots) = x^3 G(x)$

(ii) $H(x) = a_0 + a_1 3 \cdot x + a_2 3^2 \cdot x^2 + a_3 3^3 \cdot x^3 + \ldots$

$= G(3x)$

16. [**4pts**] Solve the recurrence relation

$$9a_n = 6a_{n-1} - a_{n-2}$$

when $a_0 = 6$ and $a_1 = 5$.

$9a_n = 6a_{n-1} - a_{n-2} \iff a_n = \frac{6}{9} a_{n-1} - \frac{1}{9} a_{n-2}$

Characteristic eqn: $r^2 - \frac{6}{9}r + \frac{1}{9} = (r - 1/3)^2$. Root $r = 1/3$

Sol'n has form $a_n = \alpha_1 (1/3)^n + \alpha_2 \cdot n (1/3)^n$

Solve for α_1, α_2

(1) $a_0 = \alpha_1 (1/3)^0 + \alpha_2 \cdot 0 (1/3)^0 = \alpha_1 = 6$

(2) $a_1 = \alpha_1 (1/3)^1 + \alpha_2 \cdot 1 (1/3)^1 = 6 \cdot 1/3 + \alpha_2 \cdot 1/3 = 5$

$\frac{\alpha_2}{3} = 3 \quad \alpha_2 = 9$

$$\boxed{\therefore\ a_n = 6(1/3)^n + 9 \cdot n (1/3)^n}$$

Chapter 8 – Relations

display

17. [**6pts**] Consider the set $A = \{-2, -1, 0, 1, 2\}$ and consider the relation R on A where $R = \{(a, b) \mid a^2 = b^2\}$.

(i) What is the zero one matrix associated to R?

(ii) Determine if R is (a) reflexive, (b) symmetric, (c) antisymmetric, and (d) transitive. Justify your answers!

(i)
$$M_R = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(ii) (a) This is reflexive since M_R has 1's down diagonal

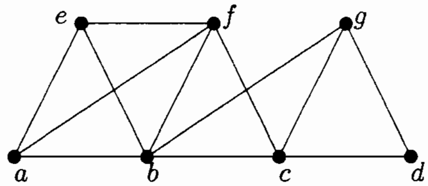
(b) This is symmetric since M_R symmetric

(c) Not antisymmetric since $(2,-2), (-2,2) \in R$

(d) This is transitive since $R^2 \subseteq R$.

Chapter 9 – Graphs

18. For the questions below, use the following graph:



(i) **[2pts]** Write out the adjacency matrix for the above graph.

(ii) **[2pts]** Find a subgraph of the above graph isomorphic to C_4.

(iii) **[2pts]** Is the above graph a planar graph? If yes, draw the graph as a planar graph; if no, explain why not.

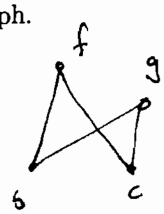
(iv) **[2pts]** Does the above graph have an Euler path? an Euler circuit? If so, write out the path/circuit.

(v) **[2pts]** Does the above graph have a Hamilton path? a Hamilton circuit? If so, write out the path/circuit.

(vi) **[2pts]** Find the chromatic number of the above graph.

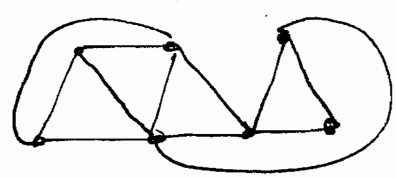
(i)
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(ii)



is a Subgp $\simeq C_4$

(iii) Planar:

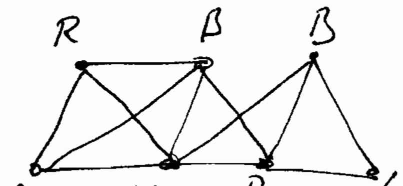


(iv) No Euler path or circuit since it has ≥ 3 vertices of odd degree

(v) Hamilton Circuit (and thus Hamilton Path)

e, a, f, c, d, g, b, e

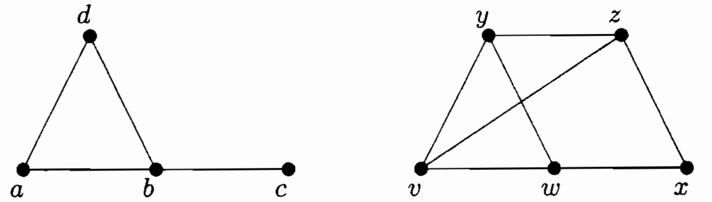
(vi) Chromatic $\# = 4$



↳ need 4 colors for this part

19. [**6pts**] We say that connected graph is **almost Eulerian** if it contains a circuit that uses every edge in the graph once and one edge in the graph twice.

(i) Determine if the following two graphs are almost Eulerian:

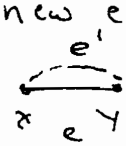


(ii) Find a characterization of *almost Eulerian* graphs similar to the characterization of Euler graphs given in class.

(i) First graph is almost Eulerian: a, b, c, b, d, a ← used edge {b, c} twice
2nd graph not almost Eulerian

(ii) G is almost Eulerian iff it has two vertices of odd degree that are adjacent.

(To see this, note that if we add a new edge e' that joins the two vertices in the edge used twice

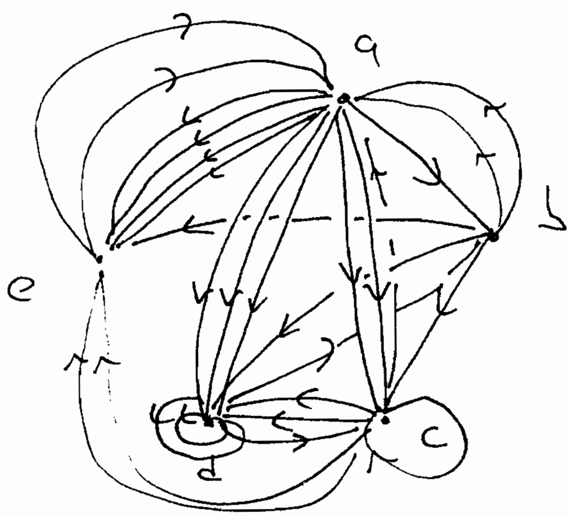


we get a ~~regular~~ Standard Eulerian graph).

20. [**2pts**] Draw the graph whose adjacency matrix has the form

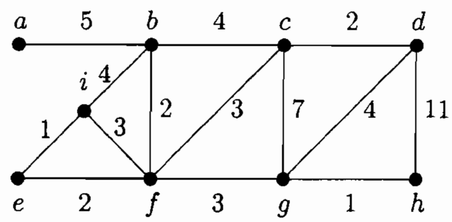
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

← not symmetric, so <u>directed</u>



Chapter 10 – Trees

21. Answer the following questions about the weighted graph given below:



(*i*) **[2pts]** Find the shortest path between *a* and *h* (you can do this by inspection).

(*ii*) **[3pts]** Use Prim's algorithm to find a minimal spanning tree of the above graph. As in class, list the order in which you picked the edges for your spanning tree, and draw your spanning tree.

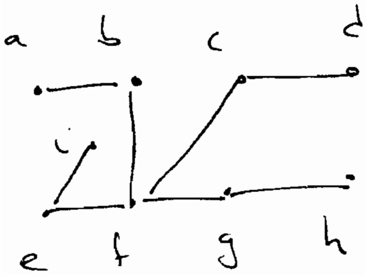
(*iii*) **[3pts]** Now adapt Kruskal's Algorithm to find a **maximal spanning tree**, i.e., a spanning tree of largest weight. As in class, list the order in which you picked the edges for your spanning tree, and draw your spanning tree.

(i) a, b, f, g, h   total distance = 5+2+3+1 = $\boxed{11}$

(ii) Prim's Alg

Choice	Edge	Weight
1	{g,h}	1
2	{f,g}	3
3	{b,f}	2
4	{e,f}	2
5	{e,i}	1
6	{f,c}	3
7	{c,d}	2
8	{a,b}	5

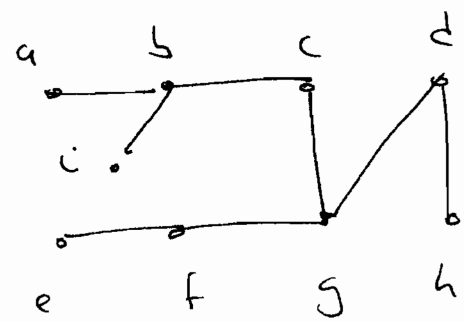
Total = 19



(iii) Pick largest possible edge

Choice	Edge	Weight
1	{d,h}	11
2	{c,g}	7
3	{a,b}	5
4	{i,b}	4
5	{b,c}	4
6	{g,d}	4
7	{f,g}	3
8	{e,f}	2

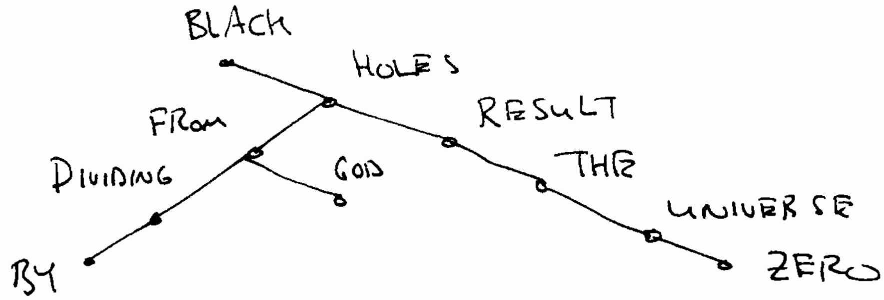
Total = 40



22. [**2pts**] Create a binary search tree for the following list of words in the quote:

Black holes result from God dividing the universe by zero.

(Assume that normal alphabetical ordering is being used.)



Chapter 11 – Boolean Algebras

23. [**2pts**] Using a table, express the values of the Boolean function:

$$F(x, y, z) = x(yz + \overline{x}\,\overline{z}).$$

x	y	z	yz	$\bar{x}\bar{z}$	$yz+\bar{x}\bar{z}$	$x(yz+\bar{x}\bar{z})$
1	1	1	1	0	1	1
1	1	0	0	0	0	0
1	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	1	1	0	1	0
0	1	0	0	1	1	0
0	0	1	0	0	0	0
0	0	0	0	1	1	0

24. In the strange world of ANTI-LAND, democracy works backwards, that is, the minority makes the decision. So, in a committee of three people, a motion will pass if 1 or less people vote for it, but the motion will fail if 2 or more people vote for it.

   (*i*) **[4pts]** Design a circuit that implements the voting of this committee using inverters, AND gates, and OR gates.

X (*iii*) **[4pts]** Can you simplify your circuit in (*i*)? If so, do so; otherwise, explain why you cannot simplify your circuit.

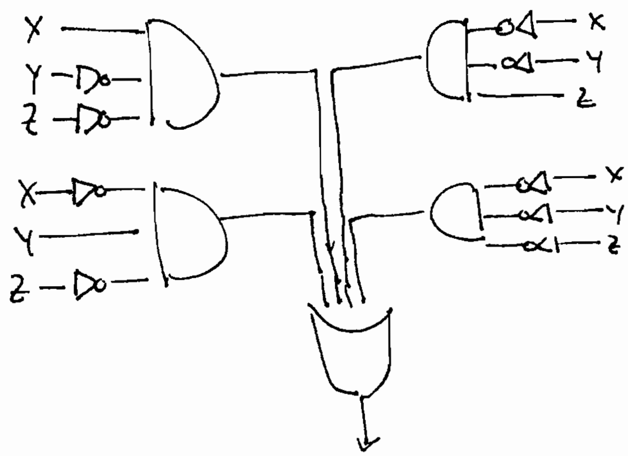
(i) Start by finding a Boolean Expression for the function.

X	Y	Z	F(X,Y,Z)
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

$F(X,Y,Z) = \cancel{XYZ} + \bar{X}\cdot Y\cdot\bar{Z} + \bar{X}\cdot\bar{Y}\cdot Z + \bar{X}\cdot\bar{Y}\cdot\bar{Z}$

$X\cdot\bar{Y}\cdot\bar{Z}$

Circuit



(ii) Simplify F using a Karnaugh map

	YZ	$Y\bar{Z}$	$\bar{Y}\bar{Z}$	$\bar{Y}Z$
X			1	
$\bar{X}$		1	1	1

$F(X,Y,Z) = \bar{Y}\bar{Z} + \bar{X}\bar{Z} + \bar{X}\bar{Y}$

Simplified circuit: