Math 2R03 (Linear Algebra II)
Due: November 22, 2024

Homework Assignment 8

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let V be a vector space over $\mathbb{C}$ with $\dim V = 6$. Suppose that $T \in \mathcal{L}(V)$ is a linear operator with eigenvalues $\lambda_1 = 2, \lambda_2 = -3$, and $\lambda_3 = 2024$. Furthermore, suppose that the multiplicity of λ_1 is 1, the multiplicity of λ_2 is 2, and the multiplicity of λ_3 is 3. What are all the possibilities for the minimal polynomial of T (justify your answer).

Hint. There are six.

Exercise 2. Let $V = \mathcal{P}_1(\mathbb{R})$, and consider the inner product on V given by

$$\langle p(x), q(x)\rangle = \int_{-2024}^{2024} p(x)q(x)\ dx.$$

With this inner product:

(1) Compute $\|x+1\|$.
(2) Find a non-zero polymomial $q(x)$ such that $\langle x+1, q(x)\rangle = 0$.

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a vector space over F with $\dim V = n$, and let $T \in \mathcal{L}(V)$. Suppose that there exists a non-zero vector $w \in V$ such that the n vectors

$$w, Tw, T^2w, \dots, T^{n-1}w$$

form a basis for V.

(1) Prove that there are $a_0, \dots, a_{n-1} \in F$ such that

$$a_0w + a_1Tw + a_2T^2w + \cdots a_{n-1}T^{n-1}w + T^nw = 0.$$

(2) With $a_0, \dots, a_{n-1}$ as in the previous part, prove that the minimal polynomial of T divides the polynomial

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + z^n.$$

Remark. Although you don't need to prove this, the above result is actually stronger. That is, you can prove that $p(z)$ *is* the minimal polynomial of T in this special case.

Exercise 4. Let $V \in \mathbb{R}^{2,2}$ be the vector space of 2×2 matrices with entries in $\mathbb{R}$. If $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$, then the trace of the matrix A, denoted trace(A), is defined as trace$(A) = a + d$. Prove that the operation

$$\langle A, B\rangle = \text{trace}(A^TB)$$

is an inner product on V. Here, A^T denotes the transpose of the matrix A.

Remark. There is nothing special about $n = 2$ in the above problem. In fact, you can show that this operation defines an inner product on $V = \mathbb{R}^{n,n}$. However, to simplify your proofs, I'm only asking you to do the special case of $n = 2$.

Additional Suggested Problems. [Not graded]

Problems 8.A - # 12, 13, 17, 21 8.B - # 2, 4, 5, 7, 9, 11, 12 6.A - # 3, 4, 5, 9, 13, 17, 21, 26