Math 2R03 (Linear Algebra II)
Due: December 2, 2024 (NOTE DATE CHANGE!)

Homework Assignment 9

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 6pts]

Exercise 1. Let $V = \mathbb{R}^3$, and consider the inner product on V given by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = 2x_1y_1 + x_2y_2 + x_3y_3.$$

Starting with the basis $w_1 = (1, 1, -1)$, $w_2 = (1, -1, 1)$, $w_3 = (-1, 1, 1)$, of $\mathbb{R}^3$, use the Gram-Schmidt Procedure to find an orthonormal basis of V with respect to this inner product.

Remark. The numbers in your answer will not be "nice"; expect square-roots! Also, read this carefully! This is not the regular dot product – notice the extra 2 at the beginning.

Exercise 2. Let $V = \mathbb{R}^3$, and consider the same inner product on V as given in Exercise 1. Consider the linear functional $\varphi : V \to \mathbb{R}$ given by

$$\varphi((x_1, x_2, x_3)) = x_1 + x_2 + x_3.$$

By the Riesz Representation Theorem, there exists a unique vector $u = (u_1, u_2, u_3) \in V$ such that

$$\varphi((x_1, x_2, x_3)) = \langle (x_1, x_2, x_3), (u_1, u_2, u_3) \rangle$$

for all $(x_1, x_2, x_3) \in V$. What is $u = (u_1, u_2, u_3)$?

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a finite dimenional vector space and $T \in \mathcal{L}(V, W)$. Show that

$$T = TP_{\mathrm{null}(T)^\perp} = P_{\mathrm{range}(T)}T.$$

Here, P_U denotes the projection map onto a subspace U.

Exercise 4. Suppose that $F = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if $\langle Tv, v \rangle = \langle T^*v, v \rangle$ for all $v \in V$.

Additional Suggested Problems. [Not graded]

Problems 6.B #2a, 3, 7, 8a, 11, 12; 6.C #1, 2, 3, 5; 7.A #1, 2, 3, 6, 7, 15; 7.B #1, 2, 10, 14