Math 2R03 (Linear Algebra II)
Due: February 6, 2021

Homework Assignment 3

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the other will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let U be a subspace of the vector space F^{2021} with $\dim U = 1010$. Suppose W is another subspace of F^{2021} such that $U + W = F^{2021}$ and $\dim U \cap W \geq 1000$. Find upper and lower bounds on $\dim W$.

Exercise 2. Fix an integer $m \geq 1$, and let $\mathrm{Int} \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}), \mathcal{P}_{m+1}(\mathbb{R}))$ be the linear map given by

$$\mathrm{Int}(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m) = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_m}{m+1}x^{m+1}.$$

Prove that $\dim \mathrm{Null}(\mathrm{Int}) = 0$.

Part B. [Proof Questions; 6pts]

Exercise 3. Suppose that $T \in \mathcal{L}(V, W)$ is injective, and that $v_1, \ldots, v_n$ are linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ are linear independent in W.

Exercise 4. Fix integers $1 \leq n \leq m$. Let $e_1, \ldots, e_n$ be the standard basis of F^n and $f_1, \ldots, f_m$ be the standard basis of F^m. For $i = 1, \ldots, n$, define the linear map $T^{(i)}$ to be the linear map

$$T^{(i)}e_k = \begin{cases} f_i & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

That is, $T^{(i)}$ is the linear map that sends e_i to f_i, and it sends all the other basis elements to 0. The elements $T^{(1)}, T^{(2)}, \ldots, T^{(n)}$ all belong to $\mathcal{L}(F^n, F^m)$ Show that these elements are linearly independent elements in the vector space $\mathcal{L}(F^n, F^m)$.

Hint. You need to show that if $c_1, \ldots, c_n$ are constants such that $c_1T^{(1)} + \cdots + c_nT^{(n)} = 0$, the zero linear map in $\mathcal{L}(F^n, F^m)$, then $c_1, \ldots, c_n$ are all zero. Note that $(c_1T^{(1)} + \cdots + c_nT^{(n)})$ is a *linear map.*