MATH 2R03 (LINEAR ALGEBRA II)
DUE: FEBRUARY 27, 2021

HOMEWORK ASSIGNMENT 4

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ be the linear transformation given by

$$Tp = (1 + 2x + 3x^2)p.$$

Using the standard bases $\{1, x, x^2\}$ for $\mathcal{P}_2(\mathbb{R})$ and $\{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4(\mathbb{R})$, find $\mathcal{M}(T)$, the matrix of T with respect to these two bases.

Exercise 2. Let $V = F^\infty$. Consider the two linear maps $T, S \in \mathcal{L}(F^\infty, F^\infty)$ where

$$T(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$$

and

$$S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots).$$

Prove that $TS = I$, but $ST \neq I$.

Part B. [Proof Questions; 7pts]

Exercise 3. Let V and W be finite dimensional vector spaces (with $\dim V \neq 0$ and $\dim W \neq 0$) and $T \in \mathcal{L}(V, W)$. Suppose that there are bases $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_m$ for W such that $\mathcal{M}(T)$ with respect to these bases contains only the values 2021, i.e.,

$$\mathcal{M}(T) = \begin{bmatrix} 2021 & \cdots & 2021 \\ \vdots & & \vdots \\ 2021 & \cdots & 2021 \end{bmatrix}.$$

Prove that $\dim \operatorname{Null}(T) = \dim V - 1$.

Exercise 4. Let V be a finite dimensional vector space, and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Remark. Note that Exercise 2 shows that this exercise is false if V is not finite dimensional.

Exercise 5. Let z_1, z_2, z_3 be three distinct complex numbers, and let w_1, w_2, w_3 be any three complex numbers (not necessarily distinct). Prove that there is a unique polynomial $p \in \mathcal{P}_2(\mathbb{C})$ such that $p(z_i) = w_i$ for $i = 1, \ldots, 3$.

Hint. Set up a system of linear equations, and use the fact from Math 1B03 that if A is a square matrix, $Az = w$ has a unique solution if and only if the determinant of A is non-zero. You may also wish to expand out $(z_3 - z_2)(z_3 - z_1)(z_2 - z_1)$.

Remark. There is nothing special about 3 in the above exercise. The above result is still true if we take n distinct complex numbers $z_1, \ldots, z_n$ and any n-tuple $(w_1, \ldots, w_n)$ of complex numbers. To prove this result, you need a result about Vandemonde matrices.