Math 2R03 (Linear Algebra II)
Due: March 6, 2021

Homework Assignment 5

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $T \in \mathcal{L}(V)$ and suppose that U and W are subspaces of V that are invariant under T. Prove that $U \cap W$ is also invariant under T.

Exercise 2. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be given by

$$T(x, y, z) = (0, 2021z, -x).$$

Find all the eigenvalues and corresponding eigenvectors.

Part B. [Proof Questions; 6pts]

Exercise 3. Consider the linear operator $T \in \mathcal{L}(F^\infty)$ given by

$$T(z_1, z_2, z_3, z_4, z_5, \ldots) = (z_2, z_4, z_6, z_8, \ldots).$$

Prove that for each positive integer $n \in \mathbb{N}$, the integer n is an eigenvalue of T.

Remark. We proved (Theorem 5.13) that if $\dim V = n$, then any $T \in \mathcal{L}(V)$ has at most n distinct eigenvalues. This result show that if V is infinite dimensional, we can have infinitely many distinct eigenvalues.

Exercise 4. Suppose V is a finite dimensional vector space. Suppose $T \in \mathcal{L}(V)$ is a linear operator with eigenvalue λ, and let v be an eigenvector corresponding λ. Let $p(z) = 3 + 2021z + 2z^2$, and consider the linear operator $p(T)$. Prove that $p(\lambda)$ is an eigenvalue of $p(T)$ with corresponding eigenvector v.