Math 2R03 (Linear Algebra II)
Due: March 27, 2021

Homework Assignment 7

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $V = F^{2021}$. Given an example of an operator $T \in \mathcal{L}(V)$ with characteristic polynomial $p(z) = (z - 2021)^{2021}$, but minimal polynomial $q(z) = (z - 2021)^2$.

Exercise 2. Suppose V is a vector space with $\dim V = 7$. Let $T \in \mathcal{L}(V)$ be an operator which has exactly two eigenvalues, namely $\lambda_1 = 1$ and $\lambda_2 = 2021$. Prove that

$$(T - I)^6(T - 2021I)^6 = 0.$$

Part B. [Proof Questions; 7pts]

Exercise 3. Let $T \in \mathcal{L}(V)$, and suppose that there is a Jordan Basis for V such that the matrix of T with respect to this basis has the form

$$\mathcal{M}(T) = \begin{bmatrix} 2 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 2 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 3 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 3 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 3 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

where $x_1, \ldots, x_5$ are unknown numbers. Suppose it is also known that the minimal polynomial of T is $p(z) = (z - 2)^2(z - 3)$. What are the values of x_1, x_2, x_3, x_4, x_5?

Exercise 4. Let a_0, a_1, a_2 be any complex numbers. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be the operator

$$T(x, y, z) = (-a_0 z, x - a_1 z, y - a_2 z).$$

(1) Show that $p(T) = 0$ where

$$p(z) = z^3 + a_2 z^2 + a_1 z + a_0.$$

(2) Show that the minimal polynomial of T cannot have degree ≤ 2. Hint: look at the vectors $(1, 0, 0), T(1, 0, 0)$ and $T^2(1, 0, 0)$.

(3) Use the previous two parts to argue that $p(z)$ is the minimal polynomial of T.

Remark. See Queston 8.C #18 for a generalization.