Math 2R03 (Linear Algebra II)
Due: April 3, 2021

Homework Assignment 8

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $V = \mathbb{R}^3$, and consider the inner product on V given by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 + 2x_3y_3.$$

Starting with the basis $w_1 = (1, 1, -1)$, $w_2 = (1, -1, 1)$, $w_3 = (-1, 1, 1)$, of $\mathbb{R}^3$, use the Gram-Schmidt Procedure to find an orthonormal basis of V with respect to this inner product.

Remark. The numbers in your answer will not be "nice"; expect lots of square-roots!

Exercise 2. Let $V = \mathbb{R}^3$, and consider the same inner product on V as given in Exercise 1. Consider the linear functional $\varphi : V \to \mathbb{R}$ given by

$$\varphi((x_1, x_2, x_3)) = x_1 + x_2 + x_3.$$

By the Riesz Representation Theorem, there exists a unique vector $u = (u_1, u_2, u_3) \in V$ such that

$$\varphi((x_1, x_2, x_3)) = \langle (x_1, x_2, x_3), (u_1, u_2, u_3) \rangle$$

for all $(x_1, x_2, x_3) \in V$. What is $u = (u_1, u_2, u_3)$?

Part B. [Proof Questions; 6pts]

Exercise 3. Let $V \in \mathbb{R}^{2,2}$ be the vector space of 2×2 matrices with entries in $\mathbb{R}$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the trace of the matrix A, denoted $\text{trace}(A)$, is defined as $\text{trace}(A) = a + d$. Prove that operation

$$\langle A, B \rangle = \text{trace}(A^T B)$$

is an inner product on V. Here, A^T denotes the transpose of the matrix A.

Remark. There is nothing special about $n = 2$ in the above problem. In fact, you can show that this operation defines an inner product on $V = \mathbb{R}^{n,n}$. However, to simplify your proofs, I'm only asking you to do the special case of $n = 2$.

Exercise 4. On the cover of your textbook is Apollonius's Identity. Prove this identity. (This is Exercise 6.A #31.)

Hint. In $V = \mathbb{R}^2$ with the standard Euclidean inner product, $\|v\|$ is the length of the vector $v \in \mathbb{R}^2$. This fact was mentioned in Math 1B03.