MATH 2R03 (THEORY OF LINEAR ALGEBRA)
DUE: JANUARY 29, 2022

HOMEWORK ASSIGNMENT 2

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the other will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Suppose that $V = \text{span}(v_1, v_2, v_3)$. Prove that the list

$$v_1 + v_2 + v_3, v_2, 2022v_3$$

also spans V.

Exercise 2. Let $V = \mathbb{C}^2$. Note that V is a vector space over $\mathbb{R}$ if we use the scalar operation

$$r(a + bi, c + di) = (ra + (rb)i, rc + (rd)i)$$

for any $r \in \mathbb{R}$ and $(a + bi, c + di) \in \mathbb{C}^2$. At the same time, V is also a vector space over $\mathbb{C}$ if we use the scalar operation

$$z(a + bi, c + di) = (z(a + bi), z(c + di))$$

for any $z \in \mathbb{C}$ and $(a + bi, c + di) \in \mathbb{C}^2$. Show that if we view V as a vector space over $\mathbb{C}$, then the vectors $(1, i), (i, -1) \in V$ are linearly dependent, but if we view V as a vector space over $\mathbb{R}$, then $(1, i), (i, -1) \in V$ are linearly independent.

Part B. [Proof Questions; 6pts]

Exercise 3. Let $p_1(x) = 3x + x^3$ and $p_2(x) = 2022$ be elements in the vector space $V = \mathcal{P}_3(\mathbb{R})$. Extend $\{p_1(x), p_2(x)\}$ to a basis of V.

Exercise 4. Prove that the functions $\sin x, \sin 2x$, and $\sin 3x$ are linearly independent on the interval $[0, \pi]$.

Hint. Assume $c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x = 0$. To show that $c_i = 0$, multiply through by $\sin ix$, and integrate over the interval $[0, \pi]$. The following identity will also be useful:

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)].$$

Remark. The above result can be extended to show that $\sin x, \sin 2x, \ldots, \sin mx$ are linearly independent on $[0, \pi]$ for any integer $m \geq 1$. The fact that these functions are linearly independent plays an important role in Fourier Series.

Additional Suggested Problems. [Not graded]

Problems 1.C # 19, 20, 2.A # 8, 9, 10