Math 2R03 (Theory of Linear Algebra)
Due: February 5, 2022

Homework Assignment 3

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the other will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let U be a subspace of the finite dimensional vector space V, i.e., $U \subseteq V$. Prove that if $\dim U = \dim V$, then $U = V$.

Remark. This result is very useful to show two vector spaces are equal. We use this fact throughout the course.

Exercise 2. Fix an integer $m \geq 1$, and let $\mathrm{D} \in \mathcal{L}(\mathcal{P}_{m+1}(\mathbb{R}), \mathcal{P}_m(\mathbb{R}))$ be the linear map given by

$$\mathrm{D}(a_0 + a_1x + a_2x^2 + \cdots + a_{m+1}x^{m+1}) = a_1 + 2a_2x + 3a_3x^2 + \cdots + (m+1)a_{m+1}x^m.$$

Prove that $\dim \mathrm{Null}(\mathrm{D}) = 1$.

Part B. [Proof Questions; 6pts]

Exercise 3. Let $V = \mathbb{R}^3$ and consider the subspaces $U = \mathrm{span}((1,0,0),(0,1,0))$ and $W = \mathrm{span}((2,1,0),(0,0,1))$ in V. Find a basis for $U \cap W$.

Hint. What is $\dim(U \cap W)$?

Exercise 4. Suppose that $v_1, \ldots, v_{2022} \in F^{2021}$ are 2022 distinct vectors. Prove that for any vector space W and for any linear map $T \in \mathcal{L}(F^{2021}, W)$, the vectors $Tv_1, \ldots, Tv_{2022}$ are linear dependent in W.

Additional Suggested Problems. [Not graded]

Problems 2.B #5, 6, 2.C #11, 14, 15, 3.A #4, 7