MATH 2R03 (THEORY OF LINEAR ALGEBRA)
DUE: FEBRUARY 26, 2022

HOMEWORK ASSIGNMENT 4

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $V = \mathbb{R}^2$, and consider the two bases $\mathcal{B} = \{(2,5),(3,7)\}$ and $\mathcal{C} = \{(-1,2),(1,-4)\}$ for V. Let $I \in \mathcal{L}(V,V)$ be the identity map, i.e., $Iv = v$ for all $v \in V$.

(1) Find $\mathcal{M}(I,\mathcal{B},\mathcal{C})$ and $\mathcal{M}(I,\mathcal{C},\mathcal{B})$.
(2) Verify that $\mathcal{M}(I,\mathcal{B},\mathcal{C})^{-1} = \mathcal{M}(I,\mathcal{C},\mathcal{B})$, that is, you are showing that the inverse of the matrix $\mathcal{M}(I,\mathcal{B},\mathcal{C})$ is the matrix $\mathcal{M}(I,\mathcal{C},\mathcal{B})$.

Exercise 2. Let $V,\mathcal{B}$, and $\mathcal{C}$ be as above, and let $T \in \mathcal{L}(V,V)$. Suppose that

$$T(x,y) = (2022x, 2022x + 2022y).$$

(1) Find $\mathcal{M}(T,\mathcal{B},\mathcal{B})$ and $\mathcal{M}(T,\mathcal{C},\mathcal{C})$.
(2) Prove that

$$\mathcal{M}(T,\mathcal{C},\mathcal{C}) = \mathcal{M}(I,\mathcal{B},\mathcal{C})\mathcal{M}(T,\mathcal{B},\mathcal{B})\mathcal{M}(I,\mathcal{C},\mathcal{B}).$$

Hint. You are allowed to use Matlab or Octave.

Remark. The matrix $\mathcal{M}(I,\mathcal{B},\mathcal{C})$ is called the *change of basis* matrix from $\mathcal{B}$ to $\mathcal{C}$. You may have seen this matrix in Math 1B03 or 2LA3. See Section 4.6 of Lay, Lay, and McDonald, where they use the notation $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

Part B. [Proof Questions; 7pts]

Exercise 3. Let V be a n-dimensional vector space with basis $\mathcal{B} = \{v_1,\ldots,v_n\}$. Let $T \in \mathcal{L}(V,V)$. Prove that

$$\operatorname{null}(T) = \{v \in V \mid \mathcal{M}(T,\mathcal{B},\mathcal{B})\mathcal{M}(v) = 0 \in F^n\}$$

Remark. The above result tells us that if we want to find the null space of an operator, it is enough to pick any basis $\mathcal{B}$, and then look at the null space of the corresponding matrix $\mathcal{M}(T,\mathcal{B},\mathcal{B})$.

Exercise 4. Let V be a finite dimensional vector space with $\dim(V) = n$. The *rank* of $T \in \mathcal{L}(V)$, denoted $\operatorname{rank}(T)$, is defined to be $\operatorname{rank}(T) = \dim \operatorname{range}(T)$. Suppose that $R,S \in \mathcal{L}(V)$. Show that if $\operatorname{rank}(SR) = n$, then R is invertible.

Exercise 5. Suppose that $p(z) \in \mathcal{P}(\mathbb{C})$ is a polynomial of degree ≥ 3 such that $0 = p(2022) = p'(2022) = p''(2022)$ (here, p' and p'' denote the first and second derivatives of p). Prove that $(z-2022)^3$ divides $p(z)$.

Additional Suggested Problems. [Not graded]

Problems 3.B #5, 19, 20 3.C #2, 3, 3.D #9, 13 Chap 4 #3, 4