MATH 2R03 (THEORY OF LINEAR ALGEBRA)
DUE: MARCH 5, 2022

HOMEWORK ASSIGNMENT 5

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $T \in \mathcal{L}(V)$, and suppose that λ is an eigenvalue of T. Prove that $2022 + 2022\lambda + 2022\lambda^2 + \cdots + 2022\lambda^{2022}$ is an eigenvalue of the linear operator

$$2022I + 2022T + 2022T^2 + \cdots + 2022T^{2022}.$$

Exercise 2. Suppose that $T \in \mathcal{L}(\mathbb{R}^2)$ is given by

$$T(x, y) = (x + 2022y, 2022x + y).$$

If $p(z) = 3 + 2x + 4x^2$, what is $p(T)$?

Hint. Feel free to use computer software to help you with this question.

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a vector space with $\dim V = 5$ and let $T \in \mathcal{L}(V)$. Suppose that for every subspace $W \subseteq V$ with $\dim W = 4$, the linear operator T is invariant on W. Prove that *every* non-zero vector $v \in V$ is an eigenvector of T.

Exercise 4. Let V be a vector space with $\dim V = 2022$. Suppose that $T \in \mathcal{L}(V)$ has the property that $\dim \operatorname{Null}(T) = 1000$. Prove that T has at most 1023 distinct eigenvalues.

Hint. Why is 0 an eigenvalue of T? Once you show this, you then have to show that T has at most 1022 distinct non-zero eigenvalues.

Additional Suggested Problems. [Not graded]

Problems 5.A #7, 11, 14, 21 5.B #2, 3, 7

Additional Comment. You will notice that in Chapter 5 of Axler's textbook, there is minimal discussion on how to find or compute eigenvalues (the emphasis is on the existence of these values). In Math 1B03 you learned how to compute eigenvalues using the characteristic polynomial $\det(A - \lambda I_n)$ for an $n \times n$ matrix A. You can use the following procedure to find eigenvalues of $T \in \mathcal{L}(V)$:

- Find $\mathcal{M}(T)$ with respect to a basis for V. This matrix is a $n \times n$ matrix with $n = \dim V$.
- Compute the eigenvalues of the matrix $\mathcal{M}(T)$ as you would in Math 1B03.
- The eigenvalues of $\mathcal{M}(T)$ are the eigenvalues of the operator T.

Note that there are some things that need to proved: (a) this procedure works regardless of what basis you picked, and (b) the eigenvalues of $\mathcal{M}(T)$ and T are the same. These details are covered in Chapter 10, which we will not cover. However, you can assume this result.