Math 2R03 (Theory of Linear Algebra)
Due: March 12, 2022

Homework Assignment 6

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $V = \mathbb{R}^3$. Give an example of an operator $T \in \mathcal{L}(V)$ that has two eigenvalues $\lambda_1 = 2022$ and $\lambda_2 = -1$, but that is not diagonalizable. Justify your answer.

Exercise 2. A particular operator $T \in \mathcal{L}(\mathbb{C}^3)$ is known to have two eigenvalues, $\lambda_1 = 2022$ and $\lambda_2 = 12$. It is also known that the multiplicity of $\lambda_2 = 12$ is 1. By Theorem 8.29, there is a basis so $\mathbb{C}^3$ so that $\mathcal{M}(T)$ can be represented by a block diagonal matrix. Determine as many values of this block diagonal matrix as you can from the given information.

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a vector space with $T \in \mathcal{L}(V)$. Suppose that λ is an eigenvector of T. The *index* of a generalized eigenvector v of λ is the smallest integer f such that $v \in \operatorname{null}((T - \lambda I)^f)$.

Suppose that w_f is a generalized eigenvector of T of eigenvalue λ with index f. Prove that there exists generalized eigenvectors $w_1, w_2, \dots, w_{f-1}$ of λ such that

(1) w_i has index i;
(2) $(T - \lambda I)w_i = w_{i-1}$ for $i = 2, \dots, f$, and
(3) $(T - \lambda I)w_1 = 0$.

Exercise 4. Suppose that V is a complex vector space with $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct <u>non-zero</u> eigenvalues of T. Prove that

$$\dim G(\lambda_1, T) + \cdots + \dim G(\lambda_m, T) \leq \dim \operatorname{range}(T).$$

Hint. The operator T may or may not have zero as an eigenvalue. Consider both cases.

Additional Suggested Problems. [Not graded]

Problems 5.C # 3, 5, 9 8.A # 2, 3, 15 8.B # 1, 2, 4