Math 2R03 (Theory of Linear Algebra)
Due: March 26, 2022

Homework Assignment 7

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

For this assignment, you may use the following lemma without proof.

Lemma 0.1. *Let B a $d \times d$ matrix of the form*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $B, B^2, B^3, \ldots, B^{d-1} \neq 0$, but $B^d = 0$.

Part A. [Short Questions; 4pts]

Exercise 1. Let V be a vector space over $\mathbb{C}$ with $\dim V = 5$. Suppose that $T \in \mathcal{L}(V)$ is a linear operator with eigenvalues $\lambda_1 = 2, \lambda_2 = -3$, and $\lambda_3 = 2022$. Furthermore, suppose that the multiplicity of λ_1 is 1, the multiplicity of λ_2 is 2, and the multiplicity of λ_3 is 2. What are all the possibilities for the minimal polynomial of T (justify your answer).

Hint. There are four.

Exercise 2. Let $V = F^4$ for a field F. Give an example of an operator $T \in \mathcal{L}(V)$ with characteristic polynomial $(z - 2022)^4$ but minimal polynomial $(z - 2022)^2$.

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a vector space over F with $\dim V = n$, and let $T \in \mathcal{L}(V)$. Suppose that there exists a non-zero vector $w \in V$ such that the n vectors

$$w, Tw, T^2 w, \ldots, T^{n-1} w$$

form a basis for V.

(1) Prove that there are $a_0, \ldots, a_{n-1} \in F$ such that

$$a_0 w + a_1 T w + a_2 T^2 w + \cdots a_{n-1} T^{n-1} w + T^n w = 0.$$

(2) With $a_0, \ldots, a_{n-1}$ as in the previous part, prove that the minimal polynomial of T divides the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n.$$

Remark. Although you don't need to prove this, the above result is actually stronger. That is, you can prove that $p(z)$ *is* the minimal polynomial of T in this special case.

Exercise 4. Let V be a finite dimensional vector space over $F = \mathbb{C}$. Suppose that $T \in \mathcal{L}(V)$ has charactertistic polynomial $(z-2022)^n$ and minimal polynomial $(z-2022)^m$ with $1 \leq m < n$. By Theorem 8.60, there is a Jordan Basis of V so that the matrix of T with respect to this basis has the form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & A_p \end{bmatrix}$$

where each A_j is a $d_j \times d_j$ upper triangular matrix of the form

$$A_j = \begin{bmatrix} 2022 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2022 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2022 & 1 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 2022 \end{bmatrix}.$$

Show for all $j = 1, \ldots, p$, we have $d_j \leq m$. Then show that there is at least one j such that $d_j = m$. The lemma given at the beginning will help.

Hint. To start this problem, what does the matrix of $T - 2022I$ look like with respect to this basis?

Remark. The point of this exercise is to show that the minimal polynomial controls the size of the blocks in the Jordan form. This result generalizes to the case that T has other eigenvalues; you are just looking at the special case that T has one eigenvalue.

Additional Suggested Problems. [Not graded]

Problems 8.C # 1, 3, 8, 13 8.D # 3, 5