MATH 2R03 (THEORY OF LINEAR ALGEBRA)
DUE: APRIL 2, 2022

HOMEWORK ASSIGNMENT 8

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $V = \mathcal{P}_1(\mathbb{R})$, and consider the inner product on V given by

$$\langle p(x), q(x)\rangle = \int_{-2022}^{2022} p(x)q(x)\ dx.$$

Find an orthonormal basis of V with respect to this inner product.

Exercise 2. Let $V = \mathcal{P}_1(\mathbb{R})$, and consider the same inner product on V as given in Exercise 1. Consider the linear functional $\varphi : V \to \mathbb{R}$ give by

$$\varphi(a_0 + a_1 x) = a_0 + a_1.$$

By the Reisz Representation Theorem, there exists a unique $q(x) \in V$ such that

$$\varphi(p(x)) = \langle p(x), q(x)\rangle$$

for all $p(x) \in V$. What is $q(x)$?

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be an inner product space and suppose that $S, T \in \mathcal{L}(V)$. Prove that $S = T$ if and only if $\langle Tv_1, v_2\rangle = \langle Sv_1, v_s\rangle$ for all $v_1, v_2 \in V$.

Exercise 4. Let V be a finite dimensional inner product space, and suppose that U is a subspace of V. The *orthogonal complement* of U, denoted $U^\perp$ is the set

$$U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}.$$

This is also a subspace of V (see Section 6.C for more details).

Suppose that $u_1, \ldots, u_s$ is a basis of U, and we extend this basis to a basis of V, say

$$u_1, \ldots, u_s, w_1, \ldots, w_t.$$

Prove that if the Gram-Schmidt process applied to the list $u_1, \ldots, u_s, w_1, \ldots, w_t$ returns the vectors vectors $g_1, \ldots, g_s, h_1, \ldots, h_t$, then $h_1, \ldots, h_t$ is a basis for $U^\perp$.

Hint. You may use the fact that $\dim U + \dim U^\perp = \dim V$ in a finite dimensional vector space (see Theorem 6.50).

Additional Suggested Problems. [Not graded]

Problems 6.A # 4, 8, 12, 19 6.B # 3, 5, 7, 9