Name: ______________________

Student Number: ______________________

Math 3GR3 C01 (Abstract Algebra) Final Exam

Day Class — Instructor: Adam Van Tuyl

Duration of Exam: 2.5 hours

McMaster University Final Exam — December 18, 2017

This examination paper includes 3 pages and 19 questions (Part A consists of 9 questions, Part B consists of 10 questions, and there is one bonus question). You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

Special Instruction

- Only a Casio FX-991 MS or MS Plus calculator is allowed
- All solutions should be written in an exam booklet.
- Do all the questions from Part A. For Part B, do 6 of the 10 questions. There is also one bonus question.
- The exam is out of 60 points.
- This paper must be returned with your answers.

PART A. (Do all the questions from this part)

1. **[5pts]** Give an example (without a proof) of
   (i) a group that is abelian, but not cyclic.
   (ii) a group of order 10 that is not abelian.
   (iii) a ring that is not commutative.
   (iv) a ring that has no unity (multiplicative identity).
   (v) a ring that is a integral domain, but not a field.

2. **[5pts]** Determine if the following statements are true or false (no proof needed).
   (i) All subgroups are normal.
   (ii) A cyclic group is abelian.
   (iii) $\mathbb{Z}$ is an ideal of $\mathbb{Q}$.
   (iv) Every field is an integral domain.
   (v) If R and S are fields, then $R \times S$ is a field.

3. **[2pts]** Let a, b, c, d be elements of a group G. Simplify: $(ab^2)^{-1}(c^2a^{-1})^{-1}(c^2b^2d)(ad)^{-2}a$.

4. **[4pts]** Consider the following element of S_{10}

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 10 & 6 & 8 & 9 & 7 & 5 \end{pmatrix}.$$

   (i) Rewrite the above permutation as a product of disjoint cycles.
   (ii) Compute the order of this permutation.

5. **[4pts]** Prove that every subgroup of an abelian group G is normal.

6. **[2pts]** Explain why $\mathbb{Z}_{2017}$ is an integral domain.

7. **[4pts]** Define a map $f : \mathbb{Z}_6 \to \mathbb{Z}_2$ by $f(a) = a$, i.e., the class of a in $\mathbb{Z}_6$ is sent to the class of a in $\mathbb{Z}_2$. Show
   (i) f is a ring homomorphism.
   (ii) Compute the kernel of f.

8. **[2pts]** Use the Division Algorithm to find the remainder of $3x^2 + 2x + 1$ when divided by $2x + 4$ over $\mathbb{Z}_5[x]$. (Note: the field is $\mathbb{Z}_5$, not $\mathbb{R}$!)

9. **[2pts]** Show that $p(x) = 5x^5 - 6x^4 - 3x^2 + 9x - 15$ is irreducible over $\mathbb{Q}[x]$.

PART B. (Do **six** of the following 10 problems)

1. **[5pts]** Let H be a subgroup of a group G. Show that the set
$$C(H) = \{x \mid xhx^{-1} = h \text{ for all } h \in H\}$$
is a subgroup of G.

2. **[5pts]** Show that the set
$$\mathbb{Z}[\sqrt{2}] = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$$
is a subring of the ring of real numbers $\mathbb{R}$.

3. **[5pts]** Let $G = SL(2, \mathbb{R})$ be the special linear group, i.e., the set of all invertible 2×2 matrices with entries in $\mathbb{R}$ and determinant one. Show that the set
$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$
is a normal subgroup of G. (Hint: $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{d}{ad-cb} & -\frac{b}{ad-cb} \\ -\frac{c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$)

4. **[5pts]** Suppose that I and J are ideals of a ring R. Show that $I \cap J$ is also an ideal of R.

5. **[5pts]** Let p and q be distinct primes, and suppose that G is a group with $|G| = pq$. Suppose that $f : G \to H$ is an onto group homomorphism, but not one-to-one. Prove that H is abelian. (Hint: First Isomorphism Theorem for Groups.)

6. **[5pts]** Suppose R is a field and $f : R \to S$ is a ring homomorphism. Prove that either f is the zero map, i.e., $f(r) = 0$ for all $r \in R$, or f is one-to-one.

7. **[5pts]** In class we proved the following statement: If $f : R \to S$ is an *onto* ring homomorphism, and if I is an ideal of R, then $f(I) = \{f(r) \mid r \in I\}$ is an ideal of S. Show that this statement is false if we remove the word "onto".

8. **[5pts]** Suppose $|G| = 30$. Suppose that there is a $g \in G$ such that $|g| = 6$ and the cyclic group $H = \langle g \rangle$ is a normal subgroup of G. Prove that G/H is abelian.

9. **[5pts]** (Prove Theorem 16.6) Every finite integral domain is a field.

10. **[5pts]** (Prove Proposition 17.4) Let $p(x)$ and $q(x)$ be polynomials in $R[x]$, where R is an integral domain. The $\deg p(x) + \deg q(x) = \deg(p(x)q(x))$.

Bonus **[1pt]** What was your favourite topic, and why?

END OF TEST