Math 3GR3 (Abstract Algebra)
Due: October 25, 2018

Homework Assignment 3

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

Exercise 1. Express the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 2 & 1 & 3 & 9 & 4 & 8 & 10 & 5 & 7 \end{pmatrix}$$

as a product of disjoint cycles. Now write the permutation as a product of transpositions.

Exercise 2. The group S_n is not a subgroup of S_{n+1} (since it is not even a subset of S_{n+1}). However, consider the subset

$$H = \{\sigma \in S_{n+1} \mid \sigma(n+1) = n+1\},$$

i.e., H consists of all the permutations of $\{1, \ldots, n+1\}$ that leave $n+1$ fixed.

(1) Show that H is a subgroup of S_{n+1}.
(2) Show that there is a bijection between H and S_n.

Remark. By "abusing notation", we can think of H as the group S_n, so we can think of S_n as a subgroup of S_{n+1}.

Exercise 3. Show that for all $n \geq 3$, S_n is not an abelian group.

Hint. Show that S_3 is not an abelian group, and then use the previous exercise.

Exercise 4. Show that A_{14} has an element of order 45.

Exercise 5. Let G be a group, and let $H \subseteq G$ be a subgroup. Prove that $g_1H = g_2H$ if and only if $g_1 \in g_2H$.

Exercise 6. Let H and K be subgroups of a group G (so $H \cap K$ is also a subgroup of G). For any $g \in G$, show that

$$gH \cap gK = g(H \cap K).$$

Exercise 7. Let p be a prime of the form $p = 4m + 3$ for some integer m. Show that the equation $x^2 \equiv -1 \pmod{p}$ has no solution.

Hint. Use Fermat's Little Theorem.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 5 and 6. Then do question 3 of 5.5 Sage Exercises (on the web page `http://abstract.ups.edu/aata/permute-sage-exercises.html`)