Math 3GR3 (Abstract Algebra)
Due: November 8, 2018

Homework Assignment 4

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

Exercise 1. Let $\mathbb{Z}$ be the additive group of integers, and consider the proper subgroup $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\} \subseteq \mathbb{Z}$. Show that $\mathbb{Z}$ is isomorphic to $2\mathbb{Z}$. Now show that if G is a finite group, then G cannot be isomorphic to a proper subgroup of G.

Exercise 2. Show that the additive group $\mathbb{R}$ is not isomorphic to the multiplicative group $\mathbb{R}^{\star}$.

Hint. What elements in each group have a finite order?

Exercise 3. Suppose that $G \cong H$. Show that if G has a subgroup $S \subseteq G$ with $|S| = m$, then H also has a subgroup $T \subseteq H$ with $|T| = m$.

Exercise 4. We define an **automorphism** of a group G to be an isomorphism from G to itself, i.e., there is an isomorphism $\phi : G \to G$. Prove that the operation of complex conjugation, i.e., $\phi : \mathbb{C} \to \mathbb{C}$ defined by $\phi(a + ib) = a - bi$, is an automorphism of the additive group $\mathbb{C}$.

Exercise 5. Let $N \subseteq G$ and $K \subseteq G$ be normal subgroups of a group G. Prove that the subgroup $N \cap K$ is also a normal subgroup of G.

Exercise 6. Let $G = \mathbb{Z}_{12}$ and $H = \{0, 3, 6, 9\}$. Compute the Cayley table for the group G/H.

Exercise 7. Suppose that K is a normal subgroup of G and $[G : K] = m$. Show that $g^m \in K$ for all $g \in G$.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 9 and 10. Let $\mathrm{Aut}(G)$ denote the set of all the automorphisms of a group G (as defined above). Use Sage to find $\mathrm{Aut}(\mathbb{Z}_5)$. Note that I am looking for more than just the output of Sage. I want you to explictly write out the isomorphsims. For example, if we were looking at automorphisms of $\mathbb{Z}_7$, then one automorphism is $\phi : \mathbb{Z}_7 \to \mathbb{Z}_7$ defined by $\phi(t) = 3t \pmod 7$.

Also, make a conjecture for the number of elements of $\mathrm{Aut}(\mathbb{Z}_n)$ for all $n \geq 2$. (Have you seen these numbers anywhere before?) When you write up your conjecture, make sure you formulate it as a statement. For example, you want something like: "For all $n \geq 2$, the number of elements of $\mathrm{Aut}(\mathbb{Z}_n)$ is $\underline{\text{(your answer)}}$."

Hint. It takes a little work to get the desired information. Here's an example you can use:

```
from sage.groups.abelian_gps.abelian_group_gap import AbelianGroupGap
from sage.groups.abelian_gps.abelian_aut import AbelianGroupAutomorphismGroup
G = AbelianGroupGap([6])
print G.list()
H= G.automorphism_group()
H.list()
```

The first two lines import packages needed for the computation. In the next line, I inputted the cyclic group $\mathbb{Z}_6$. I asked SAGE to output the list of elements of G. It returns

```
(1, f1, f2, f1*f2, f2^2, f1*f2^2)
```

Although this looks strange, it really is $\mathbb{Z}_6$. Recall that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. So, the `f1` refers to $(1,0)$ and `f2` is $(0,1)$. (Convince yourself that this is the same group.) We set `H` to be the set of automorphisms of G. The last command outputs the list of automorphisms, which is given by

```
([ f1*f2, f2 ] -> [ f1*f2, f2^2 ], 1)
```

There are two automorphisms here. The `1` refers to the identity isomorphism, while the first element refers to the automorphisms you obtain by sending `f1*f2`$= (1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ to `f1*f2`, and `f2` to `f2^2`$= (0,2)$. (Convince yourself that is an automorphism).