Math 3GR3 (Abstract Algebra)
Due: November 26, 2018 (Note date! This is a Monday)

Homework Assignment 5

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

Exercise 1. Let R be ring, and let $a, b \in R$. Show $(a+b)(a-b) = a^2 - b^2$ in R if and only if $ab = ba$ in R.

Exercise 2. Let R be a ring with identity 1. Suppose that $a \in R$ satisfies $a^2 = a$. Show that $1-2a$ is a unit in R.

Exercise 3. Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ where $i^2 = -1$. Show that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

Remark. This ring is called the *ring of Gaussian integers.*

Exercise 4. Let $R = \mathbb{Z}[i]$ (as defined above) and consider the principal ideal $I = \langle i \rangle$. Describe the distinct cosets of R/I. Make sure to provide a justification for your answer.

Exercise 5. Let $R = M_n(\mathbb{R})$ be the ring of $n \times n$ matrices with entries in $\mathbb{R}$ and $n \geq 2$. Show that R is not an integeral domain.

Exercise 6. Let $R = M_2(\mathbb{R})$ be the ring of 2×2 matrices with entries in $\mathbb{R}$. Let

$$I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Show that if $A, B \in I$, then $A - B \in I$, and if $A \in I$ and $M \in R$, then $AM \in I$. Explain why this is not enough to show that I is an ideal of R.

Remark. The set I is called a *right ideal* of R since it has the absorption property only when you multiply on the right. You can also have the notion of a *left ideal.*

Exercise 7. Let R be a commutative ring with identity. Suppose that I, J are ideals in R. Show that $I \cap J$ is also an ideal of R. Then, give an example an example to show that $I \cup J$ may fail to be an ideal.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapter 16. Take a look at Exercise 2 located here:

`http://abstract.ups.edu/aata/rings-sage-exercises.html`.

For this exercise, we want to do something similar for the ring $\mathbb{Z}[i]$. In particular, use SAGE to investigate ideals generated by two elements in $\mathbb{Z}[i]$. Your goal is to come up with a conjecture on what these ideals "look like". Specifically, you should have a statement of the form:

Conjecture. Let $a + bi, c + di \in \mathbb{Z}[i]$, and let

$$I = \{k_1(a + bi) + k_2(c + di) \mid k_1, k_s \in \mathbb{Z}[i]\}$$

be the ideal generated by the two elements $a + bi$ and $c + di$. Then <u>*YOUR CONCLUSIONS ABOUT THE IDEAL I*</u>*.*

Then explain how you used SAGE to come up with your answer.

To get you started, here is some code:

```
R = ZZ[I]
R.ideal(2,3+4*I)
```

The output will be

```
Fractional ideal (1)
```

This just means that the ideal $\{2a + (3 + 4i)b \mid a, b \in \mathbb{Z}[i]\}$ is the same ideal as the principal ideal $\langle 1 \rangle$. For another example,

```
R = ZZ[I]
R.ideal(4,6+8*I)
```

The output will be

```
Fractional ideal (2)
```

This just means that the ideal you inputted is the same as the principal ideal $\langle 2 \rangle$.

Remark. You do not need to prove your conjectue! The point of the exercise is to experiment with SAGE to find possible true statements.