Math 3GR3 (Abstract Algebra)
Due: September 20, 2019

Homework Assignment 1

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark.

Exercise 1. Given two sets A and B, the *symmetric difference* of A and B, denote $A\Delta B$, is the set

$$A\Delta B = \{x \mid x \in A \text{ or } x \in B \text{ but not both}\}.$$

(a) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{x \in \mathbb{R} \mid 2.5 \le x \le 6\}$, what is $A\Delta B$?
(b) Prove that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Exercise 2. The following two statements are false. Give a counterexample to each statement.
(a) Every injective function is surjective.
(b) Every surjective function is injective.
(Recall that to give a counterexample, you need to create an example to show that the given statement is false.)

Exercise 3. Let $X = \{a, b, c, d, e\}$. Give an example of a relation on X that is:
(a) reflexive, symmetric, but not transitive.
(b) reflexive, transitive, but not symmetric.
(c) symmetric, transitive, but not reflexive.

Exercise 4. Consider the set

$$R = \{(x, y) \mid x = y \text{ or } x = -y\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

(a) Prove that R is an equivalence relation on the set $\mathbb{Z}$.
(b) Describe the equivalence classes of R.

Exercise 5. Use induction to prove that $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \cdots + x^n)$ for all $n \ge 1$.

Exercise 6. Let a and b be non-negative integers, and suppose that there exists integers r and s such that $ar + bs = 1$. Show that $\gcd(a, b) = 1$.

Now give an example to show that this fact cannot be generalized. That is, show that the following statement is false: if there exists integers r and s such that $ar + bs = t > 1$, then $\gcd(a, b) = t$.

Remark. The purpose of the above exercise is to understand when the converse of Corollary 2.11 (page 21) holds.

Exercise 7. Prove that $\gcd(ab, c) = 1$ if and only if $\gcd(a, c) = 1$, and $\gcd(b, c) = 1$.

Remark. This is an "if and only if" statement, so you have two things to prove: (1) If $\gcd(ab, c) = 1$, then $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, and (2) If $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 1 and 2. Then find $\gcd(123456789, 975318642)$ and the two integers r and s such that $123456789r + 975318642s = \gcd(123456789, 975318642)$.