Math 3GR3 (Abstract Algebra)
Due: October 4, 2019

Homework Assignment 2

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark.

Exercise 1. Consider a rhombus that is not square (i.e., the four sides all have the same length, but the angles between sides is not 90°). Describe all the symmetries of the rhombus. Write down the Cayley table for the group of symmetries.

Exercise 2. Consider the set $\mathbb{Z}$ and the binary operation $\star$ on $\mathbb{Z}$ given by

$$x \star y = x + xy.$$

Does this operation make $\mathbb{Z}$ a group? Justify your answer.

Exercise 3. Find all the integers k that make $2k + 3 \equiv 8 \pmod{13}$ true.

Exercise 4. Let G be a group. Prove that if $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian.

Exercise 5. Let $G = GL_2(\mathbb{R})$, i.e., the group of 2×2 invertible matrices. (see example 3.14 in the textbook). Recall from Math 2R03 (Linear Algebra II) that we say a square matrix A is *orthogonal* if $A^{-1} = A^T$. So, orthogonal matrices belong to G since they are invertible. Let

$$O_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid A \text{ is orthogonal}\}$$

be the set of all orthogonal matrices of G. Prove that $O_2(\mathbb{R})$ is a subgroup of G.

Exercise 6. Let G be a group with subgroups K and L. Prove that $K \cap L$ is also a subgroup of G.

Exercise 7. Let p and q be distinct primes. Set $n = pq$ and let $\mathbb{T}_n$ denote the subgroup of the circle group $\mathbb{T}$ consisting of the n-th roots of unity. How many primitive nth-roots of unity does $\mathbb{T}_n$ have?

Remark. You may use the fact that $\operatorname{cis}(\frac{2k\pi}{n})$ is a primitive root if and only if $\gcd(k, n) = 1$ without proving it.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 3 and 4. Pick a prime $p \geq 3$, and consider the multiplicative groups $U(p), U(p^2), \ldots$. Use SAGE to determine which groups are cyclic. Now do the same of $p = 2$, i.e, $U(2), U(4), U(8), \ldots,$. Make a conjecture based upon your experiments. (I am asking you to come up with a statement like: "When $p \geq 3$, then the group $U(p^t)$ with $t \geq 1$ is __________. If $p = 2$, then $U(2^t)$ is __________". You are making a guess on what happens in general.)

Hint. Try to adapt the code on this page `http://abstract.ups.edu/aata/cyclic-sage-exercises.html` to the problem.

Remark. Gauss proved exactly when $U(n)$ is a cyclic group. See if you can find this result to double check your answer!