Math 3GR3 (Abstract Algebra)
Due: October 25, 2019

Homework Assignment 3

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

Exercise 1. Express the permutation (1923)(1965)(1487) as a product of disjoint cycles. Now write the permutation as a product of transpositions.

Exercise 2. The group A_n is not a subgroup of A_{n+1} (since it is not even a subset of A_{n+1}). However, consider the subset

$$H = \{\sigma \in A_{n+1} \mid \sigma(n+1) = n+1\},$$

i.e., H consists of all the even permutations of $\{1, \ldots, n+1\}$ that leave $n+1$ fixed.

(i) Show that H is a subgroup of A_{n+1}.
(ii) Show that there is a bijection between H and A_n.

Remark. By "abusing notation", we can think of H as the group A_n, so we can think of A_n as a subgroup of A_{n+1}.

Exercise 3. Show that for all $n \geq 4$, A_n is not an abelian group.

Hint. Show that A_4 is not an abelian group, and then use the previous exercise.

Exercise 4. Let p and q be distinct odd primes. Show that A_{p+q} has an element of order pq.

Exercise 5. Let G be a group, and let $H \subseteq G$ be a subgroup. Prove that $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$.

Exercise 6. Let $\mathrm{GL}_2(\mathbb{R})$ be the group of invertible 2×2 matrices with entries in $\mathbb{R}$. The subset

$$\mathrm{SL}_2(\mathbb{R}) = \{M \in \mathrm{GL}_2(\mathbb{R}) \mid \det(M) = 1\}$$

is a subgroup of $\mathrm{GL}_2(\mathbb{R})$ (you can assume this result).

(i) Let M_1, M_2 be elements of $\mathrm{GL}_2(\mathbb{R})$. Prove that the cosets $M_1(\mathrm{SL}_2(\mathbb{R})) = M_2(\mathrm{SL}_2(\mathbb{R}))$ if and only if $\det(M_1) = \det(M_2)$.
(ii) Use (i) to compute $[\mathrm{GL}_2(\mathbb{R}) : \mathrm{SL}_2(\mathbb{R})]$.

Hint. For part (i), use Exercise 5.

Exercise 7. Let p be a prime of the form $p = 4m + 3$ for some integer m. Show that the equation $x^2 \equiv -1 \pmod{p}$ has no solution.

Hint. Use Fermat's Little Theorem.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and first do the SAGE tutorials for Chapters 5 and 6.

In class, we showed that that A_4 is an example of group where the converse of Lagrange's Theorem is false, i.e., $6 | 12 = |A_4|$, but A_4 has no subgroup of order 6. Use Sage to determine what other alternating groups have this property. What do you think happens in general?