Math 3GR3 (Abstract Algebra)
Due: November 8, 2019

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## Homework Assignment 4

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

**Exercise 1**. Let $\phi : G \to H$ be an isomorphism. Prove that if $G$ is a cyclic group, then $H$ is a cyclic group.

**Exercise 2**. Show that the additive group $\mathbb{R}$ is not isomorphic to the multiplicative group $\mathbb{R}^\star$.

*Hint.* What elements in each group have a finite order?

**Exercise 3**. Let $\mathbb{T} = \{z \in \mathbb{C}^\star \mid |z| = 1\}$ be the circle subgroup of $\mathbb{C}^\star$ (see Chapter 4), and let $\mathbb{R}_{>0} = \{a \in \mathbb{R} \mid a > 0\}$. The set $\mathbb{R}_{>0}$ is a group under multiplication (you can use this fact without a proof). Prove that

$$\mathbb{C}^\star \cong \mathbb{T} \times \mathbb{R}_{>0}.$$

*Hint.* Theorem 9.27 will be useful.

**Exercise 4**. Let $H$ and $G$ be groups, and let $N$ be a normal subgroup of $H$. Prove that

$$S = \{(e_G, n) \mid n \in N\} \subseteq G \times H$$

is a normal subgroup of $G \times H$.

*Hint.* You need to show that $S$ is a subgroup, and then you need to show that it is normal.

**Exercise 5**. Let $N$ and $K$ be normal subgroups of a group $G$. Prove that the subgroup $N \cap K$ is also a normal subgroup of $G$.

*Hint.* You already proved in Assignment 2 that $N \cap K$ is a subgroup of $G$.

**Exercise 6**. Prove that if $G$ is cyclic, and if $H$ is a normal subgroup of $G$, then $G/H$ is also a cyclic group.

**Exercise 7**. An **automorphism** of a group $G$ is an isomorphism from $G$ to itself, i.e., there is an isomorphism $\phi : G \to G$. Fix a $g \in G$. Prove that the function $i_g : G \to G$ defined by $i_g(x) = g^{-1}xg$ is an automorphism of $G$.

**Exercise 8**. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 9 and 10. Let $\mathrm{Aut}(G)$ denote the set of all the automorphisms of a group $G$ (as defined above). Use Sage to find $\mathrm{Aut}(\mathbb{Z}_7)$. Note that I am looking for more than just the output of Sage. I want you to explicitly write out the isomorphisms. For example, if we were looking at automorphisms of $\mathbb{Z}_7$, then one automorphism is $\phi : \mathbb{Z}_7 \to \mathbb{Z}_7$ defined by $\phi(t) = 3t \pmod 7$.

Also, make a conjecture for the number of elements of $\mathrm{Aut}(\mathbb{Z}_n)$ for all $n \geq 2$. (Have you seen these numbers anywhere before?) When you write up your conjecture, make sure you formulate it as a statement. For example, you want something like: "For all $n \geq 2$, the number of elements of $\mathrm{Aut}(\mathbb{Z}_n)$ is <u>(your answer)</u>."

*Hint.* It takes a little work to get the desired information. Here's an example you can use:

```
from sage.groups.abelian_gps.abelian_group_gap import AbelianGroupGap
from sage.groups.abelian_gps.abelian_aut import AbelianGroupAutomorphismGroup
G = AbelianGroupGap([6])
print G.list()
H= G.automorphism_group()
H.list()
```

The first two lines import packages needed for the computation. In the next line, I inputted the cyclic group $\mathbb{Z}_6$. I asked SAGE to output the list of elements of $G$. It returns

```
(1, f1, f2, f1*f2, f2^2, f1*f2^2)
```

Although this looks strange, it really is $\mathbb{Z}_6$. Recall that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. So, the `f1` refers to $(1, 0)$ and `f2` is $(0, 1)$. (Convince yourself that this is the same group.) We set `H` to be the set of automorphisms of $G$. The last command outputs the list of automorphisms, which is given by

```
([ f1*f2, f2 ] -> [ f1*f2, f2^2 ], 1)
```

There are two automorphisms here. The `1` refers to the identity isomorphism, while the first element refers to the automorphisms you obtain by sending `f1*f2`$= (1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ to `f1*f2`, and `f2` to `f2^2`$= (0, 2)$. (Convince yourself that is an automorphism).