Math 3GR3 (Abstract Algebra)
Due: November 29, 2019

Homework Assignment 5

Do all of the questions. Three to four questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each).

Exercise 1. Prove that

$$\frac{\mathrm{GL}_2(\mathbb{R})}{\mathrm{SL}_2(\mathbb{R})} \cong \mathbb{R}^\star.$$

Recall that $\mathrm{GL}_2(\mathbb{R})$ is the group of 2×2 invertible matrices, and $\mathrm{SL}_2(\mathbb{R})$ is the group of 2×2 invertible matrices with determinant is 1.

Hint. Show that the function $\theta : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\star$ given by $\theta(A) = \det(A)$ is an onto group homomorphism.

Exercise 2. Let R be a ring with identity 1. Suppose that $0 \neq a \in R$ satisfies $a^2 = a$. Show that for $r \in R$, there exists a positive integer n such that $[(1-a)ra]^n = 0$. What is the smallest possible value of n that works for all $r \in R$?

Exercise 3. Let R be an integral domain, and suppose that $S_1 \subseteq R$ and $S_2 \subseteq R$ are subrings of R that are also integral domains. Prove that $S_1 \cap S_2$ is a subring of R that is a integral domain.

Hint. You need to first prove that $S_1 \cap S_2$ is a subring, and then prove it is an integral domain.

Exercise 4. Let R be a commutative ring with identity. Suppose that I, J are ideals in R. Show that $I \cap J$ is also an ideal of R. Then, give an example an example to show that $I \cup J$ may fail to be an ideal.

Exercise 5. Suppose that P_1 and P_2 are prime ideals in a ring R. Is the ideal $P_1 \cap P_2$ a prime ideal of R?

Exercise 6. Let R be a commutative ring, and fix an element $a \in R$. Prove that the set

$$I_a = \{x \in R \mid ax = 0\}$$

is an ideal of R.

Exercise 7. Let $R = \mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$. Fix an $s \in \mathbb{R}$ and let $\phi_s : \mathbb{R}[x] \longrightarrow \mathbb{R}$ be the evaluation homomorphism, i.e.,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto f(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0.$$

Prove that $\ker\phi_s = \langle x - s \rangle$, the ideal generated by the polynomial $x - s$.

Hint. To show $\ker\phi_s \subseteq \langle x - s \rangle$, the division algorithm for polynomials will be helpful.

Exercise 8. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapter 16 and look at the exercises located here:

`http://abstract.ups.edu/aata/rings-sage-exercises.html`.

For this exercise, we want to determine the prime ideals of $\mathbb{Z}_n$. Your goal is to come up with a conjecture on what these ideals "look like". Specifically, you should have a statement of the form

Conjecture. Consider the ring $\mathbb{Z}_n$ with $n \geq 1$. Then the prime ideals of $\mathbb{Z}_n$ are the ideals: YOUR CONCLUSIONS ABOUT THE IDEAL PRIME IDEALS

Then explain how you used SAGE to come up with your answer. You may also use the following fact without proof:

FACT. Every ideal of $\mathbb{Z}_n$ is principal.

To get you started, here is some code:

```
R = Integers(6)
I = R.ideal(2)
I.is_prime()
```

The output will be

```
True
```

This just means that the ideal $\{2a \mid a \in \mathbb{Z}_6\}$ is a prime ideal of $\mathbb{Z}_6$.

Remark. You do not need to prove your conjecture! The point of the exercise is to experiment with SAGE to find possible true statements.