MATH 4GR3 (GROUPS AND RINGS)
DUE: FEBUARY 9, 2024

HOMEWORK ASSIGNMENT 2

Do all of the questions. Three questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark (a link will be sent).

When submitting to Crowdmark, *submit a pdf version* of your solution instead of a picture. I suggest using a scanner app on your phone.

Exercise 1. Show that D_{2024} is a solvable group (for clarity, this is the dihedral group on the 2024-gon, so this group has 4048 elements).

Exercise 2. Let N be a normal subgroup of G. If N and G/N are solvable groups, show that G is a solvable group.

Exercise 3. Let $X = \{1, 2, 3, 4, 5, 6\}$ and consider the group $H = \{(1), (14)(25)(36)\}$. The elements of H act on X as functions (similar to Example 14.2 in the textbook). Determine all the unique orbits of this action and write X as partition of these orbits.

Exercise 4. Find the class equation for D_5. Show all your work.

Exercise 5. A flag with seven horizontal stripes can be coloured with three different colours. How many distinct flags can you make?

Exercise 6. What does the First Sylow Theorem tell you about all the groups of order 2024? What does the Third Sylow Theorem tell you about the Sylow 23-subgroups of a group of order 2024?

Hint. $2024 = 2^3 \cdot 11 \cdot 23$.

Exercise 7. Prove that a noncyclic group of order 21 must have 14 elements of order 3.

Hint. The theorem given below will be helpful.

Exercise 8. Prove that if G is a group with $|G| = 99$, then $G \cong \mathbb{Z}_{99}$ or $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{33}$.

Hint. Use the same hint as above.

Exercise 9. Go to `http://abstract.ups.edu/aata/aata.html` and review the tutorial in Chapter 14. Also, review the Sage documentation found here:

`https://doc.sagemath.org/html/en/thematic_tutorials/group_theory.html#conjugacy`

Now find the class equations for $D_3, D_4, \ldots, D_{10}$. (Note, you will be able to check your answer for Exercise 4).

The following theorem will be useful.

Theorem 1. *Let G be a finite group and suppose that M and N are normal subgroups of G such that $M \cap N = \{e\}$ and $|M||N| = |G|$. Then $G \cong M \times N$.*

Proof. It is enough to show tha G is an internal direct product of M and N and then apply Theorem 9.2 (all references to 2022 edition of Judson's book)

We need to check the three criteria of the internal direct products on page 153. We are given that $M \cap N = \{e\}$. We now show $G = MN$. It is immediate that $MN \subseteq G$. Since both sets are finite, it is enough to show that $|MN| = |G|$. Recall that $MN = \{mn \mid m \in M,\ n \in N\}$. We claim that all of these products are unique. Indeed, suppose that

$$m_1 n_1 = m_2 n_2$$

Then $m_2^{-1} m_1 = n_2 n_1^{-1} \in M \cap N = \{e\}$. So $n_2 n_1^{-1} = e$ and $m_2^{-1} m_1 = e$, or equivalently, $n_1 = n_2$ and $m_1 = m_2$. So, we have $|MN| = |M||N|$. Thus, by the hypotheses, $|MN| = |M||N| = |G|$, thus giving us $G = MN$.

Finally, we show $mn = nm$ for all $m \in M$ and $n \in N$. Since N is normal, $mnm^{-1} \in mNm^{-1} \subseteq N$, and thus there exists $n_1 \in N$ such that $mnm^{-1} = n_1$. Similarly, since M is normal, there exists $m_1 \in M$ such that $n^{-1}mn = m_1$. So $n_1 m = mn = nm_1$, or equivalently, $mm_1^{-1} = n_1^{-1} n \in M \cap N = \{e\}$. So, $m = m_1$ and $n = n_1$. This means $mnm^{-1} = n$, or equivalently, $mn = nm$. This now completes the proof. $\square$