Math 4GR3 (Groups and Rings)
Due: March 8, 2024

Homework Assignment 3

Do all of the questions. Three questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark (a link will be sent).

When submitting to Crowdmark, *submit a pdf version* of your solution instead of a picture. I suggest using a scanner app on your phone.

Exercise 1. Let $\mathbf{x} = (x_1, x_2, x_3)^t$ be a point on the unit sphere in $\mathbb{R}^3$, that is, $x_1^2 + x_2^2 + x_3^3 = 1$. Prove that if $A \in O(3)$, where $O(3)$ is the group of 3×3 orthogonal matrices, then $A\mathbf{x}$ is also on the unit sphere.

Exercise 2. Let $\varphi : R \to S$ be a homomorphism of rings. If J is an ideal of S and $I = \{r \in R \mid \varphi(r) \in J\}$, then prove that I is an ideal of R, and $\ker(\varphi) \subseteq I$.

Exercise 3. Let T be the set of rational numbers whose denominators (in lowest terms) are not divisible by 101 (which is a prime number). Prove that T is a subring of $\mathbb{Q}$.

Exercise 4. (This question extends the previous question.) Let I be the elements of T such that 101 divides the numerator of an element of T. Prove that I is an ideal of T, and that $T/I \cong \mathbb{Z}_{101}$.

Hint. Recall that if $101 \nmid s$, then $s \not\equiv 0$ in $\mathbb{Z}_{101}$. Furthermore, since $\mathbb{Z}_{101}$ is a field, there is a $u \in \mathbb{Z}_{101}$ such that $su = 1$ in $\mathbb{Z}_{101}$.

Exercise 5. Prove or disprove the following statements:

(1) If R is a commutative ring, then $R[x]$ is a commutative ring.
(2) If R has an identity, then $R[x]$ has an identity.
(3) If R is a field, then $R[x]$ is a field.

Exercise 6. Consider the *derivative map* $D : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

Is D a ring homomorphism? Either prove this statement or give a counterexample.

Exercise 7. Recall that an element $0 \neq a \in R$ is nilpotent if there is a positive integer $k \geq 2$ such that $a^k = 0$. Suppose that a_0 is a unit and a_1 is a nilpotent element of R. Prove that $a_0 + a_1x$ is a unit in $R[x]$.

Exercise 8. Suppose R is an integral domain. Assume that the division algorithm holds in $R[x]$. Prove that R is a field.

Exercise 9. Go to `http://abstract.ups.edu/aata/aata.html` and review the tutorial in Chapter 17. Also look for documentation on the SAGE command `quo_rem`. Use SAGE to answer all of the problems of Exercise 3 of Section 17.5