Math 4GR3 (Groups and Rings)
Due: April 8, 2024

Homework Assignment 5

Do all of the questions. Three questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark (a link will be sent).

When submitting to Crowdmark, *submit a pdf version* of your solution instead of a picture. I suggest using a scanner app on your phone.

Exercise 1. Suppose that $p(x)$ is an irreducible polynomial of degree 2024 in $\mathbb{Z}_2[x]$. How many elements are in the field $\mathbb{Z}_2[x]/(p(x))$? How does your answer change if $p(x)$ is an irreducible polynomial in $\mathbb{Z}_p[x]$ with p a prime?

Exercise 2. Show that $\sqrt{1+\sqrt{1+\sqrt{2022}}}$ is algebraic over $\mathbb{Q}$. What is the minimal polynomial of this element?

Exercise 3. If r and s are nonzero, prove that $\mathbb{Q}(\sqrt{r}) = \mathbb{Q}(\sqrt{s})$ if and only if $r = t^2 s$ for some $t \in \mathbb{Q}$.

Exercise 4. Show that $\mathbb{C}$ is algebraic over $\mathbb{R}$.

Exercise 5. Let α be an algebraic element of E over F whose minimal polynomial in $F[x]$ has odd degree. Prove that $F(\alpha) = F(\alpha^2)$.

Hint. Verify that $F(\alpha, \alpha^2) = [F(\alpha^2)](\alpha) = F(\alpha)$.

Exercise 6. Let $n_1, \dots, n_t$ be t distinct positive integers. Prove that

$$[\mathbb{Q}(\sqrt{n_1}, \dots, \sqrt{n_t}) : \mathbb{Q}] \le 2^t.$$

Exercise 7. Compute a basis for the extension $\mathbb{Q}(\sqrt{2024}, i)$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt{2024}, i) : \mathbb{Q}]$?

Exercise 8. Prove or disprove: $\mathbb{Q}(\sqrt{5}) \cong \mathbb{Q}(\sqrt{2})$.

Exercise 9. Prove that the following three statements are equivalent:

(1) F is an algebraically closed field.
(2) Every irreducible polynomial in $F[x]$ has degree 1.
(3) Every non-constant polynomial in $F[x]$ splits in F.

Exercise 10. Is it possible to construct with a straightedge and compass an isosceles triangle of perimeter 8 and area 1?

Hint. No. You may want to use Exercise 2 of Homework 4 (I used it in my solution).