Séminaire de Mathématiques Supérieures 2025: An Introduction to Recent Trends in Commutative Algebra

Lecture Notes and Tutorials

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with notes by

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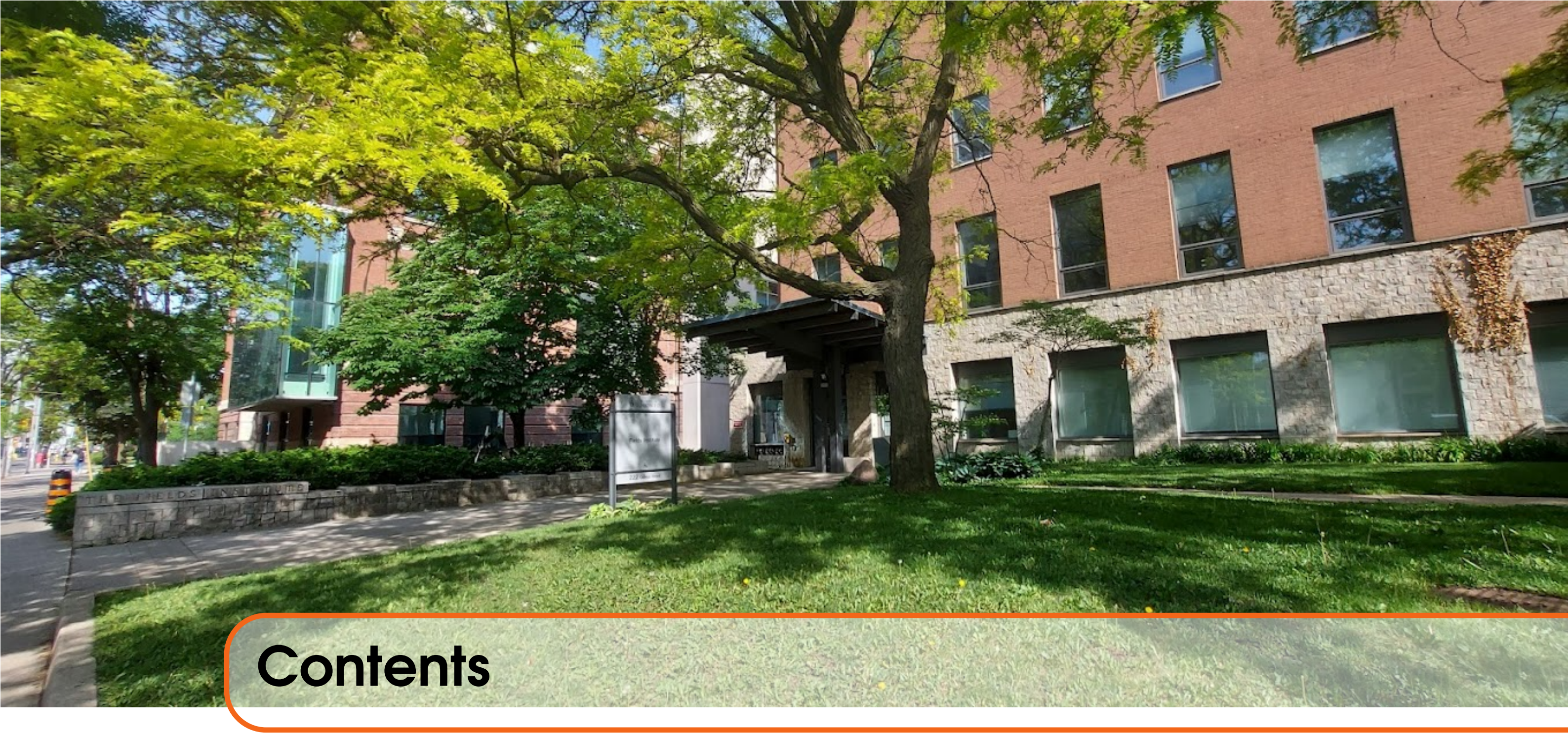

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SMS Commutative Algebra Summer School

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1. Introduction

1.1 Overview

From June 2-13, 2025, the Fields Institute hosted the *Séminaire de Mathématiques Supérieures (SMS) 2025: An Introduction to Recent Trends in Commutative Algebra*. This summer school capped off the Fields Institute's *Thematic Program in Commutative Algebra and Applications* held from Jan-June 2025.

Over the two weeks, graduate students learned about about core topics and recent advances in the field of commutative algebra. During the first week, students learned were exposed to introductory topics such as combinatorial methods in commutative algebra, computation methods in commutative algebra, characteristic p methods in commutative algebra, and homological methods in commutative algebra. For each topic, an expert in the area gave three talks and provided numerous tutorial problems for the students to enhance their understanding of the material. The second week had a similar schedule, but the topics built upon the material of the first week to talk about recent developments in the area. Talks given by experts on multigraded modules, Gröbner geometry, Hilbert functions, and new developments in positive characteristic, along with a collection of tutorials, were the focus of this week.

In this document we have collected together the lecture notes and tutorials from the summer school. It was evident as the school progressed that the lectures provided a great introduction to some of the key topics and current research in the area. Having the lectures and tutorials in a single document would of be a great benefit to the wider community.

While it would have been nice to collect all of this material into a polished book (this was actually discussed over a couple of dinners!), we have elected to simply combine all the lecture notes. What you will find are the instructors notes for their lectures. In some cases, the lecture notes are quite polished, while in some cases, we have simply included the instructor's handwritten notes. For some of the lectures, we have simply included some notes provided by students in the audience. As an aside, most of the lectures can viewed on the YouTube channel of Fields if you want to watch the original talks.

Figure 1.1: The organizers: Sergio Da Silva, Megumi Harada, Fred Galetto, Adam Van Tuyl, Elena Guardo, Patricia Klein (l. to r.)

We want to stress that these are *lecture notes*, and as such, are not polished and thoroughly proofread. This document can be used to learn about some exciting areas of commutative algebra. We encourage you to cite the original sources if you need any of the facts presented in this document, instead of citing this document.

We wish to extend a special thank you to many of the people who made this workshop a success. We first would like to thank all the speakers: Christine Berkesch, Sergio Da Silva, Sara Faridi, Federico Galetto, Elena Guardo, Jack Jeffries, Patricia Klein, Claudia Miller, and Adam Van Tuyl. The instructors not only gave great talks, but they were happy to share their material for this document. We would also like to thank Anna Brosowsky, Lauren Cranton Heller, Janet Page, and Henry Potts-Rubin for their help as TAs and Faculty Advisor during the workshop. We would also like to thank the Fields Institute in Toronto, Canada and their staff for their help. We also would like to thank the following organizations for their financial support: Combinatorial Commutative Algebra in Canada, Centre de Reserches Mathématiques, Fields Institute, Institut des sciences mathématiques, PIMS, SLMath, and the Tutte Institute. Finally, a thank you to all the students who made this a great experience.

Sergio Da Silva,
Fred Galetto,
Elena Guardo,
Megumi Harada,
Patricia Klein,
Adam Van Tuyl
Organizers of the SMS Workshop
July 2025

1.2 School Schedule

Here is a copy of the schedule of the summer school. (ChatGPT was used to convert the schedule as given on the Fields website into usable Latex code – errors may be present!)

Date	**Time**	**Event**	**Speaker**
Monday, June 2nd, 2025	09:00–09:30	Registration	
	09:30–10:20	Characteristic p Methods in Commutative Algebra (Talk)	Jack Jeffries, University of Nebraska-Lincoln
	10:30–10:45	Coffee Break	
	10:45–12:15	Characteristic p Methods in Commutative Algebra (Problem session 1)	Jack Jeffries, University of Nebraska-Lincoln
	12:15–14:15	Lunch (on your own)	
	14:15–15:05	Homological Methods in Commutative Algebra (Talk 1)	Claudia Miller, Syracuse University
	15:15–15:30	Coffee Break	
	15:30–17:00	Homological Methods in Commutative Algebra (Problem session 1)	Claudia Miller, Syracuse University
	17:30	Prenup Pub	
Tuesday, June 3rd, 2025	09:30–10:20	Computational Methods in Commutative Algebra (Talk 1)	Federico Galetto, Cleveland State University
	10:30–10:45	Coffee Break	
	10:45–12:15	Computational Methods in Commutative Algebra (Problem session 1)	Federico Galetto, Cleveland State University
	12:15–14:15	Lunch (on your own)	
	14:15–15:00	Homological Methods in Commutative Algebra (Talk 2)	Claudia Miller, Syracuse University
	15:00–15:15	Coffee Break	
	15:15–16:00	Characteristic p Methods in Commutative Algebra (Talk 2)	Jack Jeffries, University of Nebraska-Lincoln
	16:00–16:10	Break	
Wednesday, June 4th, 2025	09:30–10:20	Combinatorial Methods in Commutative Algebra (Talk 1)	Sara Faridi, Dalhousie University
	10:30–10:45	Coffee Break	
	10:45–12:15	Combinatorial Methods in Commutative Algebra (Problem session 1)	Sara Faridi, Dalhousie University
	12:15	Free Afternoon	

Thursday, June 5th, 2025	09:30–10:20	Characteristic p Methods in Commutative Algebra (Talk 3)	Jack Jeffries, University of Nebraska-Lincoln
	10:30–10:45	Coffee Break	
	10:45–12:15	Characteristic p Methods in Commutative Algebra (Problem session 2)	Jack Jeffries, University of Nebraska-Lincoln
	12:15–14:15	Lunch (on your own)	
	14:15–15:05	Homological Methods in Commutative Algebra (Talk 3)	Claudia Miller, Syracuse University
	15:15–15:30	Coffee Break	
	15:30–17:00	Homological Methods in Commutative Algebra (Problem session 2)	Claudia Miller, Syracuse University
Friday, June 6th, 2025	09:30–10:20	Combinatorial Methods in Commutative Algebra (Talk 2)	Sara Faridi, Dalhousie University
	10:30–10:45	Coffee Break	Group Photos
	10:45–12:15	Combinatorial Methods in Commutative Algebra (Problem session 2)	Sara Faridi, Dalhousie University
	12:15–14:15	Lunch (on your own)	
	14:15–15:05	Computational Methods in Commutative Algebra (Talk 3)	Federico Galetto, Cleveland State University
	15:15–15:30	Coffee Break	
	15:30–17:00	Computational Methods in Commutative Algebra (Problem session 2)	Federico Galetto, Cleveland State University
Monday, June 9th, 2025	09:30–10:20	Homological Invariants of Points in Projective Space (Talk 1)	Adam Van Tuyl, McMaster University
	10:30–10:45	Coffee Break	
	10:45–12:15	Homological Invariants of Points in Projective Space (Problem session)	Elena Guardo, Università di Catania, Adam Van Tuyl, McMaster University
	12:15–14:15	Lunch (on your own)	
	14:15–15:05	Multigraded Modules (Talk 1)	Christine Berkesch, University of Minnesota
	15:15–15:30	Coffee Break	
	15:30–17:00	Multigraded Modules (Problem session)	Christine Berkesch, University of Minnesota

Tuesday, June 10th, 2025	09:30–10:20	Gröbner Geometry and Applications (Talk 1)	Sergio Da Silva, Virginia State University
	10:30–10:45	Coffee Break	
	10:45–12:15	Gröbner Geometry and Applications (Problem session)	Sergio Da Silva, Virginia State University, Patricia Klein, Texas A&M University
	12:15–14:15	Lunch (on your own)	
	14:15–15:05	New Developments in Positive Characteristic Commutative Algebra (Talk 1)	Daniel Hernández, University of Kansas
	15:15–15:30	Coffee Break	
	15:30–17:00	New Developments in Positive Characteristic Commutative Algebra (Problem session)	Daniel Hernández, University of Kansas
Wednesday, June 11th, 2025	09:30–10:20	Homological Invariants of Points in Projective Space (Talk 2)	Elena Guardo, Università di Catania
	10:00–10:15	Coffee Break	
	10:15–10:45	Multigraded Modules (Talk 2)	Christine Berkesch, University of Minnesota
	10:45–11:00	Break	
	11:00–11:30	Gröbner Geometry and Applications (Talk 2)	Patricia Klein, Texas A&M University
	11:30–11:45	Break	
	11:45–12:15	New Developments in Positive Characteristic Commutative Algebra (Talk 2)	Daniel Hernández, University of Kansas
	12:15	Free Afternoon	
Thursday, June 12th, 2025	09:30–10:20	Homological Invariants of Points in Projective Space (Talk 3)	Elena Guardo, Università di Catania
	10:30–10:45	Coffee Break	
	10:45–12:15	Multigraded Modules (Talk 3)	Christine Berkesch, University of Minnesota
	11:45–13:45	Lunch (on your own)	
	13:45–14:35	Free choice problem session	
	15:15–15:30	Coffee Break	
	15:30–16:30	Professional development panel	
	16:30–17:00	Optional: further professional development in small groups	
Friday, June 13th, 2025	09:30–10:20	Gröbner Geometry and Applications (Talk 3)	Sergio Da Silva, Virginia State University, Patricia Klein, Texas A&M University
	10:30–10:45	Coffee Break	
	10:45–11:35	New Developments in Positive Characteristic Commutative Algebra (Talk 3)	Daniel Hernández, University of Kansas
	11:45–13:45	Lunch (on your own)	
	13:45–15:15	Free choice problem session	

1.3 School Participants

First Name	Last Name	Institution
Sergio	Da Silva	Virginia State
Fred	Galetto	Cleveland State
Elena	Guardo	University of Catania
Megumi	Harada	McMaster University
Patricia	Klein	Texas A&M
Adam	Van Tuyl	McMaster University

Table 1.1: Organizers/Instructors

First Name	Last Name	Institution
Christine	Berkesch	University of Minnesota
Sara	Faridi	Dalhousie University
Jack	Jeffries	University of Nebraska
Claudia	Miller	Syracuse University
Daniel	Hernandez	University of Kansas

Table 1.2: Instructors

First Name	Last Name	Institution
Anna	Brosowsky	University of Nebraska
Lauren	Cranton Heller	University of Nebraska
Janet	Page	North Dakota State
Henry	Potts-Rubin	Syracuse University

Table 1.3: Faculty Mentor and TAs

First Name	**Last Name**	**Institution**
Maria	Akter	University of Alabama - Tuscaloosa
Sara	Asensio	University of Valladolid (Spain)
Paulo	Assis	Federal University of Rio de Janeiro
Isidora	Bailly-Hall	University of Minnesota
Rabeya	Basu	IISER Pune
Manav	Batavia	Purdue University
Anna	Berg-Arnold	North Dakota State University
Kieran	Bhaskara	McMaster University
Jacob	Bucciarelli	Kansas State University
Eduardo	Camps	Virginia Tech
Anna Natalie	Chlopecki	Purdue University
David	Crosby	University of Arkansas
Mike	Cummings	University of Waterloo
Caitlin	Davis	University of Wisconsin-Madison
Will	DeGroot	Dartmouth College
Erin	Delargy	Duke University
Kara	Fagerstrom	University of Nebraska-Lincoln
Julianne	Faur	University of Nebraska-Lincoln
Cole	Franklin	University of Toronto
Mario	González-Sánchez	Universidad de Valladolid
Amogh	Gupta	University of Oklahoma
Valentin	Havlovec	Graz University of Technology
Haoxi	Hu	Tulane University
Tom	Huh	Pohang Science and Technology University
Ryan	Hunter	University of Kansas
Robert	Ireland	University of Nebraska-Lincoln
Siddhant	Jajodia	University of California, Irvine
Pooja Sandeep	Joshi	Texas A&M University
Parian	KHEZERLOU	Sorbonne Université / Université Paris Cité
Illya	Kierkosz	McMaster University
Elizabeth	Kodpuak	Portland State University
Allison	Kohne	George Mason University
Stephen	Landsittel	University of Missouri
Jounglag	Lim	Clemson University
Hiram	Lopez	Virginia Tech
Dipendranath	Mahato	Tulane University
Aryaman	Maithani	University of Utah
Boyana	Martinova	University of Wisconsin - Madison
Julia	McClellan	Queen’s University
Kesavan	Mohana Sundaram	University of Nebraska Lincoln
Benjamin	Mudrak	Purdue University
Emma	Naguit	McMaster University
Emma	Pickard	University of Kentucky
Ana	Podariu	University of Nebraska-Lincoln
Henry	Potts-Rubin	Syracuse University

Table 1.4: Students

First Name	**Last Name**	**Institution**
Naveena	Ragunathan	McMaster University
Neelarnab	Raha	The Pennsylvania State University
Peter	Ramsey	Louisiana State University
Johnny	Rivera, Jr.	Virginia Polytechnic Institute and State University
Giorgio Maria	Rizzo	Università degli Studi di Catania
Colleen	Robichaux	University of California, Los Angeles
Sharon	Robins	Simon Fraser University
Sandra Maria	Sandoval Gomez	University of Notre Dame
Alex	Scheffelin	Columbia University
Giovanni	Secreti	New Mexico State University
Aniketh	Sivakumar	Tulane University
Kian	Soares Da Costa	Dalhousie University
Caylee	Spivey	University of Connecticut
Dalena	Vien	Bryn Mawr College
Silas	Vriend	McMaster University
Christopher	Wong	University of Kansas
Maggie	Young	University of Missouri-Kansas City
Cleve	Young	University of Nebraska
Zongpu	Zhang	Berlin Mathematical School
Albert	Zhang	University of California, Santa Cruz

Table 1.5: Students

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2. Combinatorial Methods (S. Faridi, Notes S. Landsittel)

This course covered the basics about the connection between simplicial complexes and monomial ideals via the Stanley-Reisner and facet ideal constructions. Students learned how to use this dictionary between combinatorial algebraic topology and commutative algebra. This course was be taught by Sara Faridi (Dalhousie)

2.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video

2.2 Lecture Notes and Tutorials

The following lecture notes and tutorails are bases on Sara's lectures. The typed notes were provided by Stephen Landsittel

1. INTRODUCTION

These are notes taken (verbatim or paraphrased) from a series of two lectures by Sarah Faridi at the Fields Institute in Toronto Canada in June of 2025.

We study the combinatorical and algebraic considerations involving edge and facet ideals and their related constructions in Stanley-Reisner theory. This study involves a combination of commutative algebra, combinatorics, and algebraic topology.

TABLE 1. Methods

Field Characteristic Plays a Role	Discrete (Counting)
Stanley-Reisner Theory (Hochster's Formula)	Edge Ideals
	Facet Ideals
	Clutters

1.0.1. *Background in Combinatorics.*

We define the basic objects which are useful in algebraic and topological study of simplicial complexes.

Following the standard convention in combinatorics, a *(simplicial) complex* Δ on a (finite) set V is a set of subsets of V which is closed under subsets, and contains $\emptyset$. Sets in Δ are called *faces* and maximal faces are called *facets.* The *dimension* of a face $W \in \Delta$ is $\#W - 1$ and the *dimension* of a complex Δ is the maximal dimension of its facets. The convention that the empty set has dimension -1 is useful for homology. For positive integers n we denote the set $\{1, \ldots, n\}$ by $[n]$. Say that a complex Δ is a complex on $[n]$ if every element of $[n]$ appears in a face of Δ (this convention can vary based context or purpose).

Definition 1.1. *Let Δ be a complex on $V := [n]$. For $\sigma \in \Delta$ we define the link, deletion, and star of σ (respectively)*

$$\begin{aligned} lk_\Delta(\sigma) &= \{\alpha \in \Delta \mid \sigma \cap \alpha = \emptyset,\ \alpha \cup \sigma \in \Delta\} \\ del_\Delta(\sigma) &= \{\alpha \in \Delta \mid \sigma \cap \alpha = \emptyset\} \\ st_\Delta(\sigma) &= \{\alpha \in \Delta \mid \alpha \cup \sigma \in \Delta\}. \end{aligned}$$

We see that $lk_\Delta(\sigma) = st_\Delta(\sigma) \cap del_\Delta(\sigma)$.

Notation 1.2. *Let k be a field and let n be a positive integer. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring. For a squarefree monomial ideal $I \subset R$, the Stanley-Reisner (SR) complex of I is complex*

$$\mathcal{N}(I) := \{W \subset [n] \mid x_W \notin I\}$$

of nonfaces of I. If Δ is a complex on $[n]$ then we the SR ideal of Δ is the squarefree monomial ideal

$$I_\Delta := \{x_W \mid W \notin \Delta\}$$

of nonfaces of Δ. We define the facet ideal of a squarefree monomial ideal I

$$F(I) := \langle W \subset [n] \mid x_W \text{ is a generator of } I \rangle.$$

Definition 1.3. *Let Δ be a simplicial complex on $[n]$ vertices we define the facet ideal $F(\Delta)$ of Δ to be the squarefree monomial ideal generated by the facets of Δ. That is,*

$$F(\Delta) := (x_F \mid F \in \Delta \text{ is a facet }).$$

Remark 1.4. Let G be a graph on n vertices (write $V(G) = [n]$). G is the complex whose facets are the edges in G. In fact, it is not hard to see that $F(I_G)$ is the complex G.

1.1. Vertex Covers.

Throughout this subsection (and the rest of this document), n will be a positive integer.

Notation 1.5. *Let Δ be a complex on $[n]$. We often consider the compliment of Δ*

$$\overline{\Delta} := \{\overline{W} := [n] \setminus W \mid W \in \Delta\}$$

which is the complex of compliments of Δ's faces.

Definition 1.6. *Let Δ be a complex on $[n]$ a minimal vertex cover of Δ is a subset $A \subset [n]$ which is minimal (under inclusion) intersecting every facet of Δ.*

The complex whose facets are the minimal vertex covers of Δ is denoted by Δ_M. We have $(\Delta_M)_M = \Delta$. Moreover we see fairly quickly that $\overline{(F(I))_M} = \mathcal{N}(I)$ where I is a squarefree monomial ideal.

1.2. Alexander Duals.

Definition 1.7. *Let Δ be a simplicial complex on $[n]$. We define the Alexander Dual of Δ to be the complex of complements of the nonfaces of Δ*

$$\Delta^\vee := \{\sigma \subset [n] \mid \overline{\sigma} \notin \Delta\}.$$

We see that (for any complex Δ) $(\Delta^\vee)^\vee = \Delta$. For a squarefree monomial ideal I we define the *Alexander Dual* of I as the ideal

$$I^\vee := F(F_M)$$

where $F := F(I)$. We have that $I^\vee = I_{\mathcal{N}(I)^\vee}$.

Remark 1.8. (Some properties in Stanley-Reisner theory)
Let Δ be a simplicial complex on $[n]$ and let I be a squarefree monomial ideal in R. We have the following relationships.

- (i) $F(F(I)) = I_{\mathcal{N}(I)} = I$.
- (ii) $\mathcal{N}(I)^\vee = \overline{F(I)}$.
- (iii) $\overline{(F(I))_M} = \mathcal{N}(I)$.
- (iv) $\Delta^{\vee\vee} = \Delta$. and $I^{\vee\vee} = I$

For a monomial ideal $I \subset R$ we shall denote its polarization by $I^* \subset R^*$ (where R^* depends on I). If I is already squarefree then we have a natural ring isomorphism $R^* \to R$ ($x_{i,1} \mapsto x_i$ for all i) mapping I^* to I.

Theorem 1.9. *Let $I \subset R$ be any monomial ideal. Then I has a linear resolution if and only if $E := \Delta(I^*)^\vee$ is Macaulay.*

2. HOMOLOGY

2.1. **Algebraic topology background.** Recall that (per standard combinatorical convention) we require that (abstract simplicial) complexes contain $\emptyset$ (as in $\{\emptyset\}$ is the smallest complex) and the dimension of the face $\emptyset$ is -1 by convention.

Definition 2.1. *Let Δ be a complex and let k be a field. For $i \geq -1$, we denote the k-vector space spanned by the (formal) i-dimensional faces of Δ by $C_i := C_i(\Delta, k)$. Thus, $C_{-1} = 0$. We have k-linear maps for $i \geq 1$*

$$\begin{aligned} \partial_{i+1} :\ & C_{i+1} \to C_i \\ & x_{j,1} \dots x_{j,i+2} \mapsto \sum_l (-1)^{l+1} x_{j,1} \dots \widehat{x_l} \dots x_{j,i+2} \end{aligned}$$

which maps faces to their boundaries.

Remark 2.2. Take the notation of Definition 2.1 and let $d = \dim(\Delta)$. We have that $C_i = 0$ for $i > d$ and the sequence

$$C_\bullet = \{0 \to C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

is a chain complex concentrated only in homological degrees $\{0, 1, \dots, d\}$. We recall from algebraic topology that the complex Δ has i^{th} *reduced (simplicial) homology*

$$\widetilde{H}_i(\Delta, k) := \widetilde{H}_i(\Delta) := H_i(C_\bullet) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$$

for $i = -1, \dots, d-1$.

The i^{th} homology is given by the i-dimensional holes in Δ. For instance, a simplex (by definition) has no holes, and hence no homology. On the other hand, the outer triangle

$$\Delta = \langle ab, ac, bc \rangle \subset \langle abc \rangle$$

has homology in top dimension $d = 2$.

2.2. **Cohen–Macaulayness.**

Definition 2.3. *We say that a complex Δ is Cohen–Macaulay over k if $\widetilde{H}_i(\Delta, k) = 0$ for $i < d$.*

Say that a complex Δ is *Cohen–Macaulay* if it is Cohen–Macaulay over every field.

Take a field k and a squarefree monomial ideal $I \subset R := k[x_1, \dots, x_n]$.

Definition 2.4. *We will say that the ideal I of R is Cohen–Macaulay if R/I is Cohen–Macaulay as a standard graded ring (that is, the Krull dimension* $\dim R/I$ *of the ring R/I equals depthR/I := depth$_{m/I}R/I$, where $m := R_+$).*

Theorem 2.5. *(Reisner's Criterion)*
I is Cohen–Macaulay if and only if

$$\dim_k\left(\widetilde{H}_i(lk_{\mathcal{N}(I)}(\sigma))\right) = 0$$

for $i < \dim_k(lk_{\mathcal{N}(I)}(\sigma))$ and $\sigma \in \mathcal{N}(I)$.

A complex is called *pure* if all of its facets have the same dimension.

Remark 2.6. A Cohen–Macaulay complex is pure.

Proof. The statement follows from Reisner's Criterion. □

Definition 2.7. *A complex Δ is called a homology sphere if and only if*

$$H_i(lk_\Delta(\sigma)) = \begin{cases} 1 & i = \dim lk_\Delta(\sigma) \text{ for } \sigma \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

Reisner's Criterion implies the following statement.

Corollary 2.8. *A homology sphere is Cohen–Macaulay.*

Remark 2.9. If $I \subset R$ is Cohen–Macaulay, then the Stanley-Reisner complex $\mathcal{N}(I)$ is pure, and the facet complex $F(I)$ is pure.

Remark 2.10. Suppose that Δ is a pure complex. Then all minimal vertex covers of Δ have the same size

Example 2.11. The triangle with a single whisker

$$\langle \{a, b, c\}, \{c, d\} \rangle$$

is not Cohen–Macaulay.

Theorem 2.12. *(Fröberg)*
A graph G is cochordal if and only if I_G has resolution if and only if every power of I_G has linear resolution if and only if I_G has linear quotients.

Problem 2.13. *Let $I \subset R$ be a monomial ideal. Show using Reisner's Criterion that if $\Gamma := \mathcal{N}(I)$ is Cohen–Macaulay, then Γ is pure.*

3. Stronger Properties

Recall that a complex Δ is called pure if all of its facets have equal dimension (or equivalently, cardinality).

Remark 3.1. Let I be a squarefree Cohen–Macaulay monomial ideal. Then I is unmixed (i.e. all minimal primes of I have the same height), which is equivalent to saying that all minimal vertex covers of $F(I)$ have the same size. On the other hand, if Δ is a complex, then $F(\Delta)$ is pure if and only if all of the (minimal monomial) generators of I have the same degree.

Remark 3.2. (see the exercises from Claudia's lectures) There is a (simplicial) complex Δ which is Cohen–Macaulay over every field, but is not shellable.

3.1. **Whiskering and Grafting.**

Remark 3.3. Let Δ be any complex and let Δ' be any whiskering or grafting of Δ. Then the facet ideal $F(\Delta')$ is Cohen–Macaulay.

3.2. **Vertex Decomposability.** If Δ is a complex with vertex set V and $v \in V$ appears in some facet of Δ, then we will say that v is a *vertex on* Δ. Recall that (per the usual combinatorical convention, we require that complexes contain $\emptyset$ (as, in the smallest possible complex is $\{\emptyset\}$). If Δ is a complex, we might abuse notation and write $\Delta = \emptyset$ to indicate that $\Delta = \{\emptyset\}$.

Definition 3.4. *Suppose that Δ is pure complex. We say that Δ is vertex decomposable if and only if one of the following conditions holds*

1. *Δ is a simplex or $\{\emptyset\}$*
2. *There is a vertex v on Δ such that $lk_\Delta(v)$ and $del_\Delta(v)$ are pure vertex decomposable.*

If Δ is any complex and v is any vertex on Δ satisfying (2), then we say that v is a shedding vertex of Δ.

Example 3.5. The path on 4 vertices

$$\langle \{a, b\}, \{b, c\}, \{c, d\} \rangle$$

is vertex decomposable.

Remark 3.6. Let Δ be a complex and let Δ' be a whiskering or grafting of Δ. Then $F(\Delta')$ is Cohen–Macaulay.

Question 3.7. *(open, paraphrased)*
What are the facet complex counterparts of shellability and vertex decomposability?

Let k be a field and take a squarefree monomial ideal $I \subset R := k[x_1, \ldots, x_n]$.

Theorem 3.8. *(Eagen-Reiner) I is Cohen–Macaulay if and only if*

$$I^\vee \ (= I_{\mathcal{N}(I)^\vee}) \text{ has a linear resolution.}$$

Remark 3.9. If I is Cohen–Macaulay, then it is unmixed (as in, all minimal primes of I have the same height and hence I has no embedded primes), which is equivalent to all vertex covers of $F(I)$ having the same size. On the other hand, we have for complexes Δ that $F(\Delta)$ is pure if and only if all of the minimal generators of I have the same degree.

4. COMPUTATION OF BETTI NUMBERS

Let n be a positive integer, let k be a field, and let $R = k[x_1, \ldots, x_n]$ be the polynomial ring. Let $I \subset R$ be a squarefree monomial ideal.

Definition 4.1. *I has associated multigraded Betti numbers for each monomial $m \in R$, which **we may define** using Hochster's formula*

$$\beta_{i,m} := \beta_{i,m}(R/I) := \dim_k \left(\widetilde{H}_{i-2}(lk_{\mathcal{N}(I)^\vee}(\overline{\sigma}_m)); k \right)$$

where σ_m is the set of indices i in $[n]$ such that x_i supports m (and $\overline{\sigma_m}$ is the compliment of σ_m).

Theorem 4.2. *For all i and j, we may compute the Betti number $\beta_{i,j}$ as follows*

$$\beta_{i,j}(I) = \sum_{m \in lcm(I),\ \deg(m)=j} \beta_{i,m}(I)$$

where $lcm(I)$ is the lcm lattice of I.

Using Hocster's formula we can apply topological methods (namely discrete homotopy theory, e.g. collapses) to compute $\beta_{i,m}(I)$ as the dimension (as a vector space over a given field k) of the reduced homology over k (in a suitable degree) of some topological object.

5. Exercises

5.1. Day 1 Exercises.

Problem 5.1. *Let K be a field and let $R = K[x_1, \dots, x_n]$. An ideal $I \subset R$ is called a monomial ideal if it can be generated by monomials. Suppose that $I \subset R$ is a monomial ideal. Show that there is a unique set of minimal monomial generators of I.*

Problem 5.2. *Let Δ be a complex. Show that $\Delta_{MM}(:= (\Delta_M)_M)$ equals Δ.*

Let n be a positive integer. Fix a field K and let $R = K[x_1, \dots, x_n]$ be the polynomial ring.

Problem 5.3. *Let I be a monomial ideal. Show that the compliment of the Stanley-Reisner complex of $I \subset R$ equals Δ_M where Δ is the facet complex $F(I)$ of I*

Problem 5.4. *Let Δ be a complex and consider its facet ideal $F := F(\Delta)$. Show that the minimal primes of F are generated by the minimal vertex covers of Δ.*

Problem 5.5. *Let Δ be the hollow triangle*

$$\Delta := \langle \{1, 2\}, \{2, 3\}, \{3, 1\} \rangle$$

Show that Δ is Cohen–Macaulay using Reisner's Criterion.

Problem 5.6. *Let Δ be path on 4 vertices*

$$\Delta := \langle \{1, 2\}, \{2, 3\}, \{3, 4\} \rangle$$

Show that Δ is Cohen–Macaulay using Reisner's Criterion.

Problem 5.7. *Let $F \subset R = K[x_1, x_2, x_3, x_4]$ be the facet ideal of the complex $\Delta := \langle \{1, 2\}, \{2, 3\}, \{3, 4\} \rangle$. Use Reisner's criterion to show that depth$R/F = \dim R/F$ (as in, that F is Cohen–Macaulay).*

Problem 5.8. *Let Δ be a complex. Let $\partial_i : C_i(\Delta; K) \to C_{i-1}(\Delta, K)$, $i = 0, \dots, d$ be the differentials of the simplicial homology $C_\bullet(\Delta; K)$ of Δ. Show that $\partial_i \partial_{i+1} = 0$ for $i = 0, \dots, d - 1$.*

5.2. Day 2 Exercises.

Let n be a positive integer. Fix a field K and let $R = K[x_1, \dots, x_n]$ be the polynomial ring.

Problem 5.9. *Let $I \subset R$ be a Cohen–Macaulay ideal. Show that the Stanley Reisner Complex of I is pure using Reisner's Criterion.*

Problem 5.10. *Let Δ be the solid triangle with a leaf*

$$\Delta := \langle \{1, 2, 3\}, \{1, 4\} \rangle.$$

Why is Δ not a grafted complex (i.e. why is Δ not a grafting of another complex)?

Problem 5.11. *Find a complex Δ which is not grafted but has the property that the facet ideal $F(\Delta)$ is Cohen–Macaulay.*

Problem 5.12. *(Open) Let $I \subset R$ be a squarefree monomial ideal such that the Stanley-Reisner complex of I is shellable (vertex decomposable, respectively), what does the facet complex of I look like (in each case)?*

Problem 5.13. *Which graphs are shellable? (A graph G is the complex whose facets are its edges).*

Problem 5.14. *Let Δ be path on 4 vertices*

$$\Delta := \langle \{1, 2\}, \{2, 3\}, \{3, 4\} \rangle$$

Show that Δ is vertex-decomposable.

Problem 5.15. *(fun and probably unknown) Test extensions of Fröberg's Theorem on edge ideals of graphs with linear resolution to monomial ideals generated in degree three (see Claudia's Example 4 from her June 5th lecture).*

3. Computational methods (F. Galetto)

Gröbner bases are the underlining tool used to perform computations in commutative algebra and algebraic geometry. This course introduced the basic results of Gröbner bases, and explained their importance in computational commutative algebra. Students also learned how to use computer algebra systems to compute these bases. This course was be taught by Federico Galetto (Cleveland State)

3.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

3.2 Lecture Notes and Tutorials

We have included copies of Fred's lecture notes and his tutorials, which were provided by Fred. Note that Fred's tutorial questions are within these notes.

Computational Methods

These notes are written by Federico Galetto (Cleveland State University) for the mini-course on Computational Methods in Commutative Algebra at the Séminaire de Mathématiques Supérieures (SMS) 2025: An Introduction to Recent Trends in Commutative Algebra. You can contact the author at f.galetto@csuohio.edu.

References

- Cox, Little, O'Shea - *Ideals, Varieties and Algorithms*
- Eisenbud - *An Introduction to Commutative Algebra with a View Towards Algebraic Geometry*
- Kreuzer, Robbiano - *Computational Algebra 1*
- Ene, Herzog - *Gröbner Bases in Commutative Algebra*
- Adams, Loustaunau - *An Introduction to Gröbner Bases*

Day 1

Motivational problems

Let $R = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{k}$. Let $I = \langle f_1, \dots, f_r \rangle \subseteq R$ be an ideal. We are interested in the following problems.

Ideal membership and equality

- Given $f \in R$, determine if $f \in I$.
- If $f \in I$, find $q_1, \dots, q_r \in R$ such that $f = \sum_{i=1}^{r} q_i f_i$.
- Given an ideal J of R, determine if $I = J$.

Let $\mathbb{V}(I)$ denote the vanishing locus of I in the affine space $\mathbb{A}^n_{\mathbb{k}}$. Knowing that $f \in I$ tells us that f vanishes on all points of $\mathbb{V}(I)$. Checking ideal equality is useful when the same ideal is given two different generating sets. Also, $I = J$ implies the equality of vanishing loci $\mathbb{V}(I) = \mathbb{V}(J)$ (the converse is false).

Quotient representations

- Given $f \in R$, how should we represent the coset $f + I$ in the quotient ring R/I?
- Given $f, g \in R$, determine if $f + I = g + I$.
- Find a basis of R/I as a $\mathbb{k}$-vector space.

For example, the classes in the quotient ring $\mathbb{Z}/\langle m \rangle$ can be represented by the integers $0, \dots, m-1$. Geometrically, a polynomial $f \in R$ determines a polynomial function $\mathbb{V}(I) \to \mathbb{k}$. When $f + I = g + I$, the polynomials f and g determine the same function on $\mathbb{V}(I)$. Finding bases of R/I allows us to use linear algebra to study geometric properties of $\mathbb{V}(I)$ such as dimension and degree.

Univariate case

Before tackling these problems in full generality, it useful to focus on the one variable case $\Bbbk[x]$. In this case, we can use the fact that $\Bbbk[x]$ is a Euclidean domain.

Theorem

For every $f, g \in \Bbbk[x]$ with $g \neq 0$, there exist unique $q, r \in \Bbbk[x]$ (called *quotient* and *remainder*, respectively) such that $f = qg + r$ and $r = 0$ or $\deg(r) < \deg(g)$.

Here $\deg(g)$ denotes the largest power of the variable x appearing in g with a nonzero coefficient. In other words, if $\deg(g) = d$, then

$$g = \sum_{i=1}^{d} c_i x^i$$

with $c_d \neq 0$. We call $c_d x^d$ the *leading* or *initial term* of g and we call c_d the *leading coefficient*.

The *long division algorithm* gives an effective way to construct q and r given f and g. From here, we can solve the problems above. For example, letting $I = \langle g \rangle$, we have:

- $f \in I$ if and only if $r = 0$;
- if $f \in I$, then $f = qg$ where the unique q can be found explicitly;
- $f + I = r + I$, so we can choose the remainder as the standard representative modulo I;
- assuming $\deg(g) = d$, the elements $1 + I, x + I, x^2 + I, \ldots, x^{d-1} + I$ form a $\Bbbk$-basis of $\Bbbk[x]/I$.

Monomial orderings

The first thing we do when dividing f by g is line out their terms from highest to lowest degree. A multivariate division algorithm would require a similar step, but how should we order terms of a polynomial in two or more variables?

A *monomial* in $R = \Bbbk[x_1, \ldots, x_n]$ is an element of the form $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ (note: $0 \in \mathbb{N}$). We set $|a| = a_1 + a_2 + \cdots + a_n$, so $\deg(x^a) = |a|$. Some sources say *term* instead of monomial; other sources use the word term for polynomials cm where $0 \neq c \in \Bbbk$ and m is a monomial.

Definition

A *monomial ordering* on the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ is an order (meaning reflexive, antisymmetric, and transitive) relation $<$ on the set of monomials in R satisfying the following properties.

1. It is total: for all monomials $m_1 \neq m_2$, we have $m_1 < m_2$ or $m_2 < m_1$.
2. It is compatible with multiplication: for all monomials m_1, m_2, m_3, if $m_1 < m_2$, then $m_1 m_3 < m_2 m_3$.
3. Has 1 as its minimum: for all monomials $m \neq 1$, we have $1 < m$.

Here are a few notable monomial orderings.

Example: Lexicographic Order (Lex)

We write $x^a >_{\text{Lex}} x^b$ in the lexicographic order if the first nonzero entry of the vector $a - b$ is positive. When we use different letters such as x, y, z for variables, the lexicographic order is simply the alphabetical order, so $x > y > z$. However, as a result, the lexicographic order ignores degrees, so you end up with $x > y^{100}$.

Example: Graded Lexicographic Order (GLex)

We write $x^a >_{\text{GLex}} x^b$ in the graded lexicographic order if $|a| > |b|$, or $|a| = |b|$ and $x^a >_{\text{Lex}} x^b$. Thus, the graded lexicographic order prioritizes degree, and then uses the lexicographic order to break ties.

Example: Graded Reverse Lexicographic Order (GRevLex)

We write $x^a >_{\text{GRevLex}} x^b$ in the graded reverse lexicographic order if $|a| > |b|$, or $|a| = |b|$ and the rightmost nonzero entry of the vector $a - b$ is negative. The name is related to the fact that on monomials of the same degree this is the reverse of the (graded) lexicographic order if the order of the variables is reversed.

Here is the same polynomial written from largest to smallest term in the orders above.

- Using Lex: $x^4 + x^3y^2z^4 + xy^5z^3$
- Using GLex: $x^3y^2z^4 + xy^5z^3 + x^4$
- Using GRevLex: $xy^5z^3 + x^3y^2z^4 + x^4$

Although Lex and GLex seem a little more natural and have their applications, there are practical reasons for working with GRevLex (which is the default in software like Macaulay2).

Fix a monomial ordering on $R = \Bbbk[x_1, \dots, x_n]$ and let $f \in R$.

- The largest monomial appearing with a nonzero coefficient in a polynomial f is called its *leading monomial*; we denote it $\mathrm{LM}(f)$.
- The coefficient of the leading monomial is called the *leading coefficient* of f; we denote it $\mathrm{LC}(f)$.
- The product of the leading coefficient and the leading monomial gives the *leading term* of f; we denote it $\mathrm{LT}(f)$, so we have $\mathrm{LT}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f)$.

The words monomial and term are some times interchanged in the literature; also, some sources refer to leading terms/monomials as *initial* or *head* terms/monomials.

Multivariate division

Theorem

Consider an ordered collection of polynomials $F = (f_1, \dots, f_s) \in R^s$ where $R = \Bbbk[x_1, \dots, x_n]$ and fix a monomial ordering on R. For every $f \in R$, there exist $q_1, \dots, q_s, r \in R$ such that $f = \sum_{i=1}^{s} q_i f_i + r$, and $r = 0$ or r is a $\Bbbk$-linear combination of monomials none of which are divisible by any of $\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_s)$.

The element r is known as the *normal form* of f upon division by F.

Division algorithm

To construct $q_1, \dots, q_s$ and r, we initially set them equal to 0, then proceed as follows.

1. Find the smallest i such that $\mathrm{LM}(f_i)$ divides $\mathrm{LM}(f)$, if any, then go to step 2; otherwise, go to step 3.
2. Replace q_i by $q_i + \mathrm{LT}(f)/\mathrm{LT}(f_i)$ and f by $f - (\mathrm{LT}(f)/\mathrm{LT}(f_i))f_i$, then go to step 4.
3. Replace r by $r + \mathrm{LT}(f)$ and f by $f - \mathrm{LT}(f)$, then go to step 4.
4. If $f = 0$, then stop and return $q_1, \dots, q_s, r$; otherwise, go back to step 1.

For example, suppose we want to divide $f = x^2y^2 - y^3$ by $f_1 = y^2 - x$ and $f_2 = xy - 1$ in GLex. We can write out the division algorithm using the format of long division. We highlight terms added to r.

$$\begin{array}{rrrrrrr}
 & & x^2 & -y & & & \\
 & & -1 & & & & \\
\cline{3-4}
y^2 & -x & \multicolumn{1}{|r}{x^2y^2} & -y^3 & & & \\
xy & -1 & \multicolumn{1}{|r}{x^2y^2} & -x^3 & & & \\
\cline{3-4}
 & & & x^3 & -y^3 & & \\
\cline{4-5}
 & & & & -y^3 & & \\
 & & & & -y^3 & +xy & \\
\cline{5-6}
 & & & & & -xy & \\
 & & & & & -xy & +1 \\
\cline{6-7}
 & & & & & & -1 \\
\cline{7-7}
 & & & & & & 0
\end{array}$$

Therefore, we get $q_1 = x^2 - y$, $q_2 = -1$, and $r = x^3 - 1$. However, notice what happens if we swap f_1 and f_2.

$$\begin{array}{rrrrr}
 & & xy & & \\
 & & -y & & \\
\cline{3-4}
xy & -1 & \multicolumn{1}{|r}{x^2y^2} & -y^3 & \\
y^2 & -x & \multicolumn{1}{|r}{x^2y^2} & -xy & \\
\cline{3-4}
 & & & -y^3 & +xy \\
 & & & -y^3 & +xy \\
\cline{4-5}
 & & & & 0
\end{array}$$

In this case, we get $q_1 = xy$, $q_2 = -y$, and $r = 0$; it follows that $f = xyf_2 - yf_1 \in \langle f_1, f_2 \rangle$. This shows the remainder is not uniquely determined, so it cannot be used to test ideal membership. As it turns out, the fault for this behavior is not in the remainder or in the algorithm, but in the tuple F we are dividing by.

Gröbner bases

We adopt the following working definition. We will later provide equivalent characterizations.

Definition

Let $R = \Bbbk[x_1, \dots, x_n]$ and fix a monomial ordering on R. A tuple $G = (g_1, \dots, g_s) \in R^s$ of nonzero elements is a *Gröbner basis* if for every $f \in R$ there is a unique $r \in R$ with the following properties:

- $f = \sum_{i=1}^{s} q_i g_i + r$ for some $q_1, \dots, q_s \in R$;
- $r = 0$ or no term of r is divisible by any of $\mathrm{LM}(g_1), \dots, \mathrm{LM}(g_s)$.

If $I = \langle g_1, \dots, g_s \rangle$ is the ideal generated by the elements in G, we call G a Gröbner basis of I.

The r in this definition can be computed using the division algorithm. Since R contains infinitely many elements, the definition above is hard to use in practice, so we need a different way to recognize a Gröbner basis.

ⓘ **Definition**

Consider nonzero polynomials $f, g \in R = \Bbbk[x_1, \dots, x_n]$. The *S-polynomial* of f and g is

$$S(f,g) = \frac{\operatorname{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(f)} f - \frac{\operatorname{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(g)} g$$

where lcm denotes the least common multiple.

The S-polynomial $S(f,g)$ is designed to produce a cancellation of leading terms. Notice also that $S(f,g) \in \langle f, g \rangle$.

For example, consider the polynomials $f_1 = y^2 - x$ and $f_2 = xy - 1$ in GLex. We have $\operatorname{lcm}(\mathrm{LM}(f_1), \mathrm{LM}(f_2)) = \operatorname{lcm}(y^2, xy) = xy^2$. Therefore, the S-polynomial of f_1, f_2 is

$$S(f_1, f_2) = \frac{xy^2}{y^2}(y^2 - x) - \frac{xy^2}{xy}(xy - 1) = xy^2 - x^2 - xy^2 + y = -x^2 + y.$$

Theorem (Buchberger's Criterion)

Let $R = \Bbbk[x_1, \dots, x_n]$ and fix a monomial ordering on R. A tuple $G = (g_1, \dots, g_s) \in R^s$ of nonzero elements is a Gröbner basis if and only if for all $i \neq j$ the remainder of $S(g_i, g_j)$ upon division by G is zero.

The previous computation shows that $F = (f_1, f_2)$ is not a Gröbner basis because the remainder of $S(f_1, f_2)$ upon division by F is nonzero. However, if we let $f_3 = S(f_1, f_2)$, we can use Buchberger's criterion to show that $G = (f_1, f_2, f_3)$ is a Gröbner basis of $\langle f_1, f_2 \rangle$.

Finding Gröbner bases

Now the question is: does every ideal $I \subseteq R = \Bbbk[x_1, \dots, x_n]$ admit a Gröbner basis? The answer is yes! In fact, a Gröbner basis of an ideal can be constructed using a procedure due to Bruno Buchberger.

Buchberger's Algorithm

To construct a Gröbner basis of $I = \langle f_1, \dots, f_s \rangle$, set $G = (f_1, \dots, f_s)$ and proceed as follows.

1. For each pair $\{p, q\}$ in G with $p \neq q$, compute the remainder of $S(p, q)$ upon division by G. Go to step 2.
2. If all remainders computed in step 1 are zero, stop and return G; otherwise, add all nonzero remainders to G and go back to step 1.

Buchberger's Criterion ensures that this algorithm returns a Gröbner basis. Of course, one should still prove that this algorithm terminates in a finite number of steps. The algorithm above is designed to be simple but is not very efficient; however, one can introduce several optimizations. In addition, there are other algorithms that can be used to compute Gröbner bases (Hilbert drives, Faugère's F_4, signature-based) and algorithms that convert

Gröbner bases between different monomial orders (FGLM, Gröbner walk). There are also algorithms that will compute Gröbner bases of special families of ideals, such as the Buchberger-Möller algorithm for ideals of points.

Special generation

We conclude this discussion with another property that characterizes Gröbner bases. This property will be analyzed further on Day 3.

Consider again the polynomials $f_1 = y^2 - x$ and $f_2 = xy - 1$ in GLex. We observed that

$$f = -x^2 + y = xf_1 - yf_2 \in \langle f_1, f_2 \rangle.$$

One would hope that $\mathrm{LM}(f)$ is equal to either $\mathrm{LM}(xf_1)$ or $\mathrm{LM}(yf_2)$. However, we have $\mathrm{LM}(f) = x^2$ and $\mathrm{LM}(xf_1) = \mathrm{LM}(yf_2) = xy^2$; in fact, $f = S(f_1, f_2)$ so it is designed to produce a cancellation of leading terms. This cannot occur with a Gröbner basis.

Theorem

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R. A tuple $G = (g_1, \ldots, g_s) \in R^s$ of nonzero elements is a Gröbner basis if and only if for every nonzero $f \in \langle G \rangle$ there exist $q_1, \ldots, q_s \in R$ such that $f = \sum_{i=1}^{s} q_i g_i$ and

$$\mathrm{LM}(f) = \max\{\mathrm{LM}(q_i g_i) \,|\, i \in \{1, \ldots, s\}, q_i g_i \neq 0\}$$

where the maximum is taken with respect to the chosen monomial ordering.

Thus, a Gröbner basis of an ideal I can be seen as special set of generators that satisfies the property in the theorem.

Day 1 problems

Problems 1, 3, 9, 11, 12, and 13 are easier to start with. Everyone should try at least one of the problems that ask to compute a Gröbner basis (11, 12, and 13 are more hands-on; 14 and 15 are a bit more abstract). Problem 4 is also strongly recommended as the results will be used on Day 2.

Problem 1

Show that there is only one monomial order on $\Bbbk[x]$.

Problem 2

- We say a monomial ordering $\geqslant$ is *degree compatible* if $x^a \geqslant x^b$ implies $\deg(x^a) \geqslant \deg(x^b)$. For example, GLex and GRevLex are degree compatible by definition. Show that there are exactly two degree compatible monomial orderings on $\Bbbk[x, y]$.
- Show that there is only one monomial ordering on $\Bbbk[x, y]$ such that $x > y^i$ for all $i \geqslant 2$.

For more on the classification of monomial orderings for a small number of variables see Tutorial 10 in Kreuzer, Robbiano.

Problem 3

Write in increasing order the 20 smallest monomials in $\Bbbk[x, y, z]$ equipped with Lex. Do the same for GLex and GRevLex.

Problem 4

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ and fix a monomial ordering $>$ on $\Bbbk[x_1, \ldots, x_n]$. Given monomials x^a and x^b, define $x^a >_{\mathbf{u}} x^b$ if and only if:

- $\mathbf{u} \cdot a > \mathbf{u} \cdot b$ (where $\cdot$ denotes the dot product of vectors), or
- $\mathbf{u} \cdot a = \mathbf{u} \cdot b$ and $x^a > x^b$ (in the monomial ordering fixed at the beginning).

We call $>_{\mathbf{u}}$ the *weight order* determined by $\mathbf{u}$ and $>$.

- Show that $>_{\mathbf{u}}$ is a monomial ordering.
- Assume $>$ is Lex and find $\mathbf{u}$ such that $>_u$ is GLex.
- Consider a positive integer $m \leqslant n$ and let $\mathbf{u} = (1, \ldots, 1, 0, \ldots, 0)$ with m 1's and $n - m$ 0's. Let $>$ be GRevLex. Show that $>_{\mathbf{u}}$ has the following property: any monomial in $x_1, \ldots, x_m$ is greater than all monomials in $\Bbbk[x_{m+1}, \ldots, x_n]$.

Problem 5

Let M be an $n \times n$ nonsingular matrix with integer entries and denote M^T its transpose. Given monomials $x^a, x^b \in \Bbbk[x_1, \ldots, x_n]$, define $x^a \geqslant_M x^b$ if and only if $x^{aM^T} \geqslant x^{bM^n}$ in Lex, where aM^T is the product of the row vector a with the matrix M^T and similarly for bM^T.

- Prove that $\geqslant_M$ is a total order.
- Prove that $\geqslant_M$ is a monomial order if and only if the first nonzero entry of each column of M is positive.
- Find a matrix M such that $\geqslant_M$ is Lex. Do the same for GLex and GRexLex.

Problem 6

Let $>$ be a total order compatible with multiplication on the set of monomials of $\Bbbk[x_1, \ldots, x_n]$ (see our definition of monomial ordering). Recall that a *well-ordering* is a total order such that every nonempty subset contains a least element. Show that $>$ has the monomial 1 as its minimum element if and only if it is a well-ordering. [Hint: for the $\Rightarrow$ implication, use Hilbert's Basis Theorem or Dickson's Lemma.]

Problem 7

Given monomials $x^a, x^b \in \Bbbk[x_1, \ldots, x_n]$, define $x^a \geqslant x^b$ if and only if $a = b$ or the rightmost nonzero entry of the vector $a - b$ is negative; we call this relation RevLex.

- Show that RevLex is a total order compatible with multiplication.
- Show that RevLex is not a monomial ordering.

Problem 8

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R.

- Show that $\mathrm{LM}(fg) = \mathrm{LM}(f)\,\mathrm{LM}(g)$ for all nonzero $f, g \in R$.
- Show that $\mathrm{LM}(f + g) \leqslant \max\{\mathrm{LM}(f), \mathrm{LM}(g)\}$ for all nonzero $f, g \in R$ such that $f + g \neq 0$. Show that when $\mathrm{LM}(f) \neq \mathrm{LM}(g)$ the equality is achieved.

Problem 9

This problem gives another example where the remainder of division depends on the order of the divisors. Consider $\mathbb{Q}[x, y]$ with the Lex order. Let $f = x^5 - 1$, $g_1 = -x^2 + xy^2$ and $g_2 = x^2y - y^2$.

- Divide f by the tuple (g_1, g_2).
- Divide f by the tuple (g_2, g_1).

Problem 10

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R. Consider $f, g \in R$ whose leading monomials are relatively prime, meaning that $\operatorname{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f)\,\mathrm{LM}(g)$.

- Show that $S(f, g) = pg - qf$ where $p = f - \mathrm{LT}(f)$ and $q = g - \mathrm{LT}(g)$.
- Show that $\mathrm{LM}(S(f, g)) = \max\{\mathrm{LM}(pg), \mathrm{LM}(qf)\}$.
- Deduce that the remainder of $S(f, g)$ upon division by the pair (f, g) is zero.

Problem 11

Consider $R = \mathbb{Q}[x, y]$ with the lexicographic ordering. Is the tuple $F = (y^2 - x, xy - 1)$ a Gröbner basis? If not, find a Gröbner basis of the ideal $\langle F \rangle$.

Problem 12

For a little more practice with the Buchberger algorithm, compute a Gröbner basis of the ideal $\langle 2z - x^3, y - x^2 \rangle$ in $\mathbb{Q}[x, y, z]$ with the GRevLex (or with Lex if you want to see a few more steps). What are some obvious ways to improve upon the algorithm as outlined above?

Problem 13

Here is an example where the result changes with the characteristic of the field. Find a Gröbner basis of $\langle x^2 + 1, x^2y + x - y \rangle$ in $\Bbbk[x, y]$ with GRevLex, when $\Bbbk = \mathbb{Q}$ and when $\Bbbk = \mathbb{Z}/2$.

Problem 14

Let $A = (a_{i,j})$ be an $m \times n$ matrix with entries in $\Bbbk$. Let

$$f_i = a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n$$

be the linear polynomial in $\Bbbk[x_1, \ldots, x_n]$ determined by the i-th row of A, and consider the ideal $I = \langle f_1, \ldots, f_m \rangle$. Let B be the reduced row echelon form of A and let $g_1, \ldots, g_t$ be the linear polynomials determined by the nonzero rows of B (so $t \leqslant n$). Prove that $\{g_1, \ldots, g_t\}$ is a Gröbner basis of I.

Problem 15

A binomial in $R = \Bbbk[x_1, \ldots, x_n]$ is a polynomial of the form $\alpha x^a - \beta x^b$ for some nonzero $\alpha, \beta \in \Bbbk$ and some exponent vectors $a, b \in \mathbb{N}^n$. A binomial ideal in R is an ideal that has a generating set consisting entirely of binomials.

- Show that the S-polynomial of two binomials is a binomial.
- Show that the remainder of a binomial upon division by a tuple of binomials is a binomial.
- Deduce that a binomial ideal has a Gröbner basis consisting entirely of binomials.

Day 2

Reduced Gröbner bases

If you compute Gröbner bases by hand and compare with others or with a computer, you may obtain different results.

Definition

A Gröbner basis G is called *reduced* if for all g in G:

1. $\mathrm{LC}(g) = 1$;
2. no monomial of g is divisible by the leading term of any other element of G.

Reduced Gröbner bases are important for the following reason.

Theorem

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R. Every nonzero ideal I in R has a unique reduced Gröbner basis.

If a Gröbner basis G of I is known, then it is easy to produce the reduced Gröbner basis of I by normalizing coefficients and eliminating unnecessary terms. This gives us a new method to test ideal equality.

Corollary

Two ideals I, J in R are equal if and only if they have the same reduced Gröbner basis for some (hence any) monomial ordering.

We also notice that (1) is the reduced Gröbner basis of the ideal $\langle 1 \rangle = \Bbbk[x_1, \ldots, x_n]$ in any monomial ordering. The vanishing locus in the affine space $\mathbb{A}^n_{\Bbbk}$ of the ideal $\langle 1 \rangle$ is clearly empty. Conversely, by the weak Nullstellensatz, if $\Bbbk$ is algebraically closed and $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is an ideal such that $\mathbb{V}(I) = \varnothing$, then $I = \langle 1 \rangle$. This leads to the following criterion which allows us to check when a system of polynomial equations has a solution.

Corollary

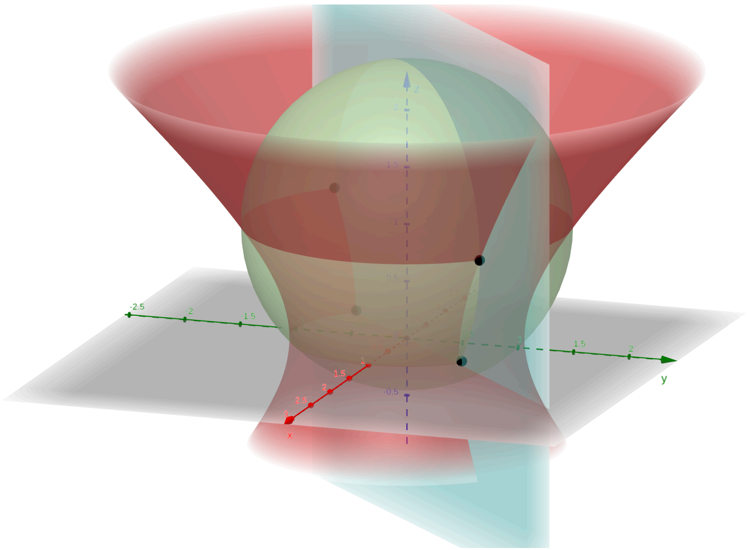
Let I be an ideal in $\Bbbk[x_1, \ldots, x_n]$ with $\Bbbk$ algebraically closed. Then $\mathbb{V}(I) = \varnothing$ if and only if the reduced Gröbner basis of I in one (hence any) monomial ordering is (1).

It is not possible to work computationally over an algebraically closed field. However, the construction of a Gröbner basis as described in Buchberger's Algorithm can be carried out over a subfield that can be represented in a computer algebra system.

Solving systems of equations

Gröbner bases may help solve systems of polynomial equations. Consider the following example, which describes the intersection of a sphere, a hyperboloid, and a plane.

$$\begin{cases} x^2 + y^2 + (z-1)^2 = 2 \\ x^2 + y^2 - z^2 = 1 \\ x = y \end{cases}$$



In Macaulay2, we set up a ring $R = \mathbb{Q}[x, y, z]$ with the lexicographic order and define the ideal

$$I = \langle x^2 + y^2 + (z-1)^2 - 2, x^2 + y^2 - z^2 - 1, x - y \rangle$$

with generators corresponding to the equations of the system.

```
R=QQ[x,y,z,MonomialOrder=>Lex]
I=ideal(x^2+y^2+(z-1)^2-2, x^2+y^2-z^2-1, x-y)
```

Observe how M2 expands all operations and arranges monomials according to the chosen ordering. Next, we compute a Gröbner basis using Macaulay2.

```
G=gb I
gens G
```

The `gb` command runs the Gröbner basis computation, then we can use `gens` to display the result as a one-row matrix. Notice that the leading terms of the elements in the Gröbner basis are arranged in increasing order: $z^2 < 2y^2 < x$. Because we chose to use Lex and the smallest leading term is a power of the smallest variable z, it follows that the other terms in the first polynomial must be smaller than z^2 and, therefore, they cannot involve other variables. Thus, we get an equivalent system

$$\begin{cases} z^2 - z = 0 \\ 2y^2 - z - 1 = 0 \\ x - y = 0 \end{cases}$$

where the first equation is univariate. This system can be solved from top to bottom by finding roots of one equation and substituting into the next. The solutions are the four points

$$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\qquad\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right),\qquad(1,1,1),\qquad(-1,-1,1).$$

In this particular example, the default ordering (GRevLex) also leads to a system with a univariate equation, but that may not always be the case.

The Gröbner basis `G` we obtained for the ideal `I` is not reduced but only because of the coefficient in $2y^2$; this choice allows M2 to avoid denominators over $\mathbb{Q}$. We can check that the paraboloid $z = x^2 + y^2 - 1$ passes through the four points by checking it belongs to `I` or, equivalently, that its remainder modulo `G` is zero.

```
f=x^2+y^2-1-z
f%G
```

To express the polynomial `f` as a linear combination of `G` we can compute the quotients of division as follows.

```
f//(gens G)
```

We can also express `f` as a linear combination of the original generators of `I`.

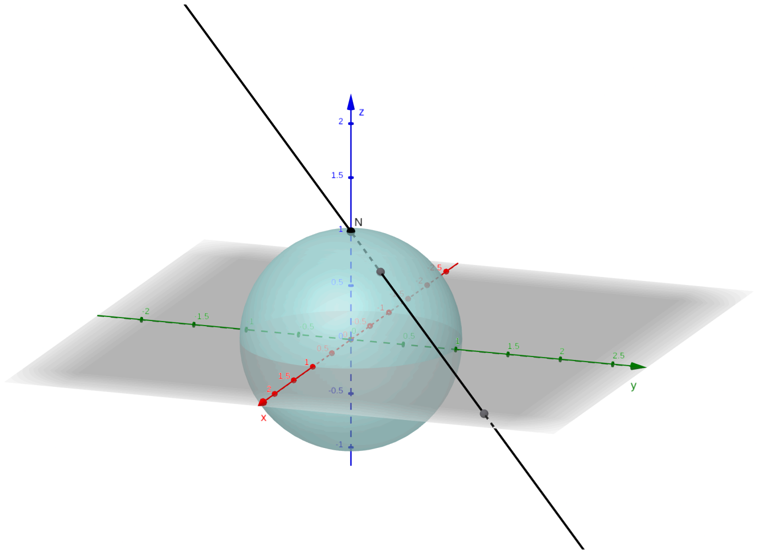
```
f//(gens I)
```

Elimination

Gröbner bases can also be used to find implicit equations for varieties parametrized by rational functions. In other words, we can use Gröbner bases to eliminate parameters. The stereographic projection from the north pole gives the rational parametrization of the sphere

$$x=\frac{2u}{1+u^2+v^2},\qquad y=\frac{2v}{1+u^2+v^2},\qquad z=\frac{-1+u^2+v^2}{1+u^2+v^2}.$$

depending on two parameters u, v.



In Macaulay2, we set up a ring $R=\mathbb{Q}[u,v,x,y,z]$ and define the ideal

$$I=\langle(1+u^2+v^2)x-2u,(1+u^2+v^2)y-2v,(1+u^2+v^2)z+1-u^2-v^2\rangle$$

with generators obtained by clearing denominators in the parametrization. We are formally interested in the so-called *elimination ideal* $I\cap\mathbb{Q}[x,y,z]$ in the subring $\mathbb{Q}[x,y,z]$. We could take the Lex order with $u>v>x>y>z$. Another option, which is typically more efficient, is to use a so-called *elimination order* designed to eliminate the first two variables u,v.

```
R=QQ[u,v,x,y,z,MonomialOrder=>Eliminate 2]
I=ideal((1+u^2+v^2)*x-2*u,
    (1+u^2+v^2)*y-2*v,
    (1+u^2+v^2)*z+1-u^2-v^2)
```

Next, we compute a Gröbner basis and display its elements.

```
G=gb I
gens G
```

The elements of this Gröbner basis involving only x,y,z give us implicit equations for the sphere. To extract these elements, we can use the command `selectInSubring`.

```
selectInSubring(1,gens G)
```

When we set up the ring with the elimination order, M2 creates two blocks of variables: u,v and x,y,z; the first argument informs M2 that we want to eliminate the variables in the first block. Another way to obtain an elimination ideal in M2 is to use the command `eliminate`.
Notice that our parametrization of the sphere misses the point $(0,0,1)$, so it only covers a subset U of the sphere which is open in the Zariski topology. The elimination ideal vanishes on the closure of U which is the whole sphere.

The ideas illustrated in this example can be generalized as follows.

ⓘ Definition

Given an ideal I in the polynomial ring $\mathbb{k}[x_1,\dots,x_n]$, the m-th *elimination ideal* I_m is the ideal of the subring $\mathbb{k}[x_{m+1},\dots,x_n]$ defined by $I_m=I\cap\mathbb{k}[x_{m+1},\dots,x_n]$.

ⓘ Definition

A monomial ordering on $\mathbb{k}[x_1,\dots,x_n]$ is of m-*elimination type* if every monomial involving one of $x_1,\dots,x_m$ is greater than all monomials in $\mathbb{k}[x_{m+1},\dots,x_n]$.

With the definitions above, we have the following result.

> **Theorem**
>
> If I is an ideal in $\Bbbk[x_1, \dots, x_n]$ and G is a Gröbner basis of I with respect to a monomial ordering of m-elimination type, then $G \cap \Bbbk[x_{m+1}, \dots, x_n]$ is a Gröbner basis of the m-th elimination ideal $I_m = I \cap \Bbbk[x_{m+1}, \dots, x_n]$.

Day 2 problems

Problem 16 is about reduced Gröbner bases and can be done by hand. The other problems showcase a variety of applications of Gröbner bases in the spirit of the Day 2 notes; use of a computer algebra system like Macaulay2 is highly recommended. Problem 21 is strongly recommended for anyone who has not seen it before.

Problem 16

If you found Gröbner bases by hand in problems 11 or 12, your results are likely not reduced. Find the reduced Gröbner bases for the ideals in those problems.

Problem 17

Consider the following system of polynomial equations.

$$\begin{cases} x^2 + y^2 + z^2 = 9 \\ 3x^2 = y^2 z \\ x^2 z + 2 = 2y^2 \end{cases}$$

How many rational, real, and complex solutions does it have?

Problem 18

A finite graph is *3-colorable* if every vertex can be assigned one of 3 different colors in such a way that vertices connected by an edge have different colors. If w denotes a primitive cubic root of unity, then we can use the complex numbers $1, w, w^2$ to represent 3 different colors. If we denote $x_1, \dots, x_n$ the vertices of our graph, assigning a color to each vertex means that each variable x_i must be assigned one of the values $1, w, w^2$. Then, the equations

$$x_i^3 - 1 = 0$$

must be satisfied for all $i \in \{1, \dots, n\}$. If x_i and x_j are connected by an edge, then $x_i \neq x_j$. Given that $x_i^3 = 1 = x_j^3$ and $x_i^3 - x_j^3 = (x_i - x_j)(x_i^2 + x_i x_j + x_j^2)$, an equation of the form

$$x_i^2 + x_i x_j + x_j^2 = 0$$

must be satisfied for each edge in the graph. It follows that the graph is 3-colorable if and only if $\mathbb{V}(I) \neq \varnothing$ where I is the ideal of $\Bbbk[x_1, \dots, x_n]$ generated by all the equations above. Now, we can use Gröbner bases to solve the following.

- Show that K_5, the complete graph on 5 vertices, is not 3-colorable.
- Let G be the graph obtained from K_5 by removing two non-incident edges. Show that G is 3-colorable.

To work over an extension of $\mathbb{Q}$ containing a primitive cubic root of unity, you can use the following Macaulay2 code. Note that $x^2 + x + 1$ is the minimal polynomial of w.

```
kk=toField( QQ[w] / ideal(w^2+w+1))
R=kk[x_1..x_5]
```

Problem 19

Shidoku is a smaller relative of Sudoku. You play on the 4×4 grid

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

and you replace each letter with an integer from 1 to 4 in a way that every row, column, and 2×2 corner block contains each of the number 1, 2, 3, and 4 exactly once. This problem shows how you can represent and solve Shidoku puzzles using Gröbner bases.

- Each letter in the grid must satisfy an equation of the form

$$(w-1)(w-2)(w-3)(w-4) = 0$$

  to ensure that it can only be equal to 1, 2, 3, or 4.
- The only way to choose four numbers w, x, y, z from the set $\{1, 2, 3, 4\}$ is for them to add up to 10 and multiply to 24; in other words, they must satisfy the equations:

$$w + x + y + z - 10 = 0, \qquad wxyz - 24 = 0.$$

- Form the ideal I in $\mathbb{Q}[a, \dots, p]$ generated by the conditions above for all variables and all choices of rows, columns, and 2×2 corner blocks. Your ideal should have 40 generators. The ideal I represents all possible Shidoku boards.
- Now, consider a particular board; for example:

			4
4		2	
	3		1
1			

  We can represent this board by adding new equations such as $d = 4$ and so on for all other values present on the board. Let J be the ideal generated by the elements of I and these new equations.
- Find a Gröbner basis of J to determine if the board above admits a unique solution. If so, use the Gröbner basis to solve the puzzle.

For more information and for more ideas on how to represent Sudoku boards algebraically, consult the article "Gröbner Basis Representations of Sudoku" by Elizabeth Arnold, Stephen Lucas, and Laura Taalman.

Problem 20

Consider the surface S in $\mathbb{R}^3$ formed by the union of all lines joining the points

$$(u^2, -u^3, u), \qquad (-u^2, u^3, 1-u)$$

for $u \in \mathbb{R}$; this is an example of a *ruled surface*.

- Write a parametrization of S.
- Use elimination to find a polynomial $f \in \mathbb{R}[x, y, z]$ such that S is contained in the set of points satisfying the implicit equation $f = 0$.

Problem 21

Consider the polynomial rings $R = \Bbbk[w, x, y, z]$ and $S = \Bbbk[s, t]$. Consider the ring homomorphism $\varphi\colon R \to S$ defined on the variables as follows:

$$\varphi(w) = s^3, \qquad \varphi(x) = s^2t, \qquad \varphi(y) = st^2, \qquad \varphi(z) = t^3.$$

The kernel of φ is the vanishing ideal of the *twisted cubic* in $\mathbb{P}^3$, an object of interest to geometers. The homomorphism φ corresponds to a parametrization of the twisted cubic, so we can use elimination to compute this kernel. Define the ideal

$$I = \langle w - s^3, x - s^2t, y - st^2, z - t^3 \rangle$$

in $\Bbbk[s, t, w, x, y, z]$.

- Show that $\ker \varphi = I \cap R$.
- Use Gröbner bases to find generators of $\ker \varphi$.

Problem 22

The trigonometric parametrization

$$\begin{cases} x = (2 + \cos(t)) \cos(u) \\ y = (2 + \cos(t)) \sin(u) \\ z = \sin(t) \end{cases}$$

describes a torus in $\mathbb{R}^3$. We show this torus lies in an affine variety by eliminating the parameters t and u to produce a polynomial equation. The trigonometric functions prevent us from using elimination directly, so set

$$a = \cos(t), \qquad b = \sin(t), \qquad c = \cos(u), \qquad d = \sin(u)$$

to replace the parametrization above with an algebraic one. However, these new variables are not independent as they must satisfy $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$. Now, form an ideal I in $\mathbb{Q}[a, b, c, d, x, y, z]$ generated by the parametrization and the relations among the new variables. Finally, use elimination to find the equation for the torus.

Problem 23

You may remember when a quadratic equation has a double root, but what about a cubic equation? Consider the polynomial $p(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \Bbbk$, where $\Bbbk$ is a field of characteristic not equal to 2 or 3, and $a \neq 0$. Recall that x_0 is a double root of $p(x)$ if and only if $(x - x_0)^2$ divides $p(x)$.

- Show that x_0 is a double root of $p(x)$ if and only if $p(x_0) = 0$ and $\frac{dp}{dx}(x_0) = 0$.
- Consider the ideal $I = \langle p, \frac{dp}{dx} \rangle$ of $\Bbbk[x, a, b, c]$. Find $I \cap \Bbbk[a, b, c, d]$ and use it to determine when p has a double root in terms of a, b, c, d.
- Similarly, find conditions on a, b, c, d guaranteeing $p(x)$ has a triple root.

Problem 24

Consider the polynomial

$$f(x,y)=y^2-(x^3+ax+b),$$

where $a,b\in\Bbbk$ and $\Bbbk$ is a field of characteristic not equal to 2 or 3. The points $(x,y)\in\Bbbk^2$ that satisfy $f(x,y)=0$ define a plane cubic curve. A point $P=(x_0,y_0)$ on this curve is called *singular* if the tangent vector at P

$$\left(\frac{\partial f}{\partial x}\bigg|_P,\frac{\partial f}{\partial y}\bigg|_P\right)$$

is zero; we say the curve is *smooth* if it has no singular points. To determine when the curve has singular points proceed as follows. Consider the ideal $I=\langle f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\rangle$ of $\Bbbk[x,y,a,b]$, then eliminate x and y to find relations between a,b. The plane cubic will be smooth, also known as an *elliptic curve*, when those relations are nonzero.

Problem 25

Fix $a\in\mathbb{C}$. The minimal polynomial of a over $\mathbb{Q}$ is the monic polynomial p with rational coefficients of the smallest degree such that $p(a)=0$, where *monic* means it has leading coefficient is 1. For example, the minimal polynomials of $a=\sqrt{2}$, $b=\sqrt[3]{5}$, and $i=\sqrt{-1}$ are, in order, a^2-2, b^3-5, and i^2+1. This problem shows how to use elimination to find the minimal polynomial of a complex number living in a particular field extension of $\mathbb{Q}$. For example, consider

$$x=\frac{b^2-i}{a}=\frac{\sqrt[3]{25}-i}{\sqrt{2}}\in\mathbb{Q}\left(\sqrt{2},\sqrt[3]{5},i\right).$$

By clearing the denominator, we obtain the algebraic relation $ax-b^2+i=0$. We take the ideal of $\mathbb{Q}[a,b,i,x]$ generated by this relation and the minimal polynomials of a, b, and i:

$$I=\langle ax-b^2+i,a^2-2,b^3-5,i^2+1\rangle.$$

Next, we use an elimination order to compute $I\cap\mathbb{Q}[x]$. Since this elimination ideal lives in $\mathbb{Q}[x]$, it can be generated by a single monic polynomial, which is the minimal polynomial of x. Find this minimal polynomial.

Problem 26

A polynomial $f\in\Bbbk[x_1,\ldots,x_n]$ is called *symmetric* if

$$f(x_1,\ldots,x_n)=f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

for every permutation σ of $\{1,\ldots,n\}$. We can use elimination orderings to identify symmetric polynomials as follows. For $1\leqslant k\leqslant n$, we define the *elementary symmetric polynomial* of degree k as

$$e_k=\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}x_{i_1}\cdots x_{i_k};$$

in other words, e_k is the sum of all squarefree monomials of degree k. In the ring $R=\Bbbk[x_1,\ldots,x_n,y_1,\ldots,y_n]$ with a monomial ordering of n-elimination type, let G be a Gröbner basis of the ideal $I=\langle e_1-y_1,\ldots,e_n-y_n\rangle$. Given $f\in\Bbbk[x_1,\ldots,x_n]\subseteq R$, let g be the remainder upon division of f by G. Then:

1. f is symmetric if and only if $g\in\Bbbk[y_1,\ldots,y_n]$;
2. if f is symmetric, then $f=g(e_1,\ldots,e_n)$ and this is the unique expression of f as a polynomial in $e_1,\ldots,e_n$.

Now, for $i \geqslant 0$, define the *power sum symmetric polynomial*

$$p_i = x_1^i + \cdots + x_n^i$$

and the *complete homogeneous symmetric polynomial*

$$h_i = \sum_{a_1+\cdots+a_n=i} x_1^{a_1} \cdots x_n^{a_n}.$$

For $n = 4$, use the ideas above to verify that $p_1, \ldots, p_4$ and $h_1, \ldots, h_4$ are symmetric, and express them as polynomials in $e_1, \ldots, e_4$.

Problem 27

Let I and J be ideals in $\Bbbk[x_1, \ldots, x_n]$.

- Show that $(tI + (1-t)J) \cap \Bbbk[x_1, \ldots, x_n] = I \cap J$. Here, tI is the ideal of $\Bbbk[t, x_1, \ldots, x_n]$ generated by $\{tf_1, \ldots, tf_r\}$ where $\{f_1, \ldots, f_r\}$ is a set of generators of I; the ideal $(1-t)J$ is constructed similarly.
- Use elimination to compute $I \cap J$ where $I = \langle x^2y - z, xy + 1\rangle$ and $I = \langle x - y, z^2 - x\rangle$ are ideals of $\Bbbk[x, y, z]$.

Problem 28

Let I and J be ideals in $R = \Bbbk[x_1, \ldots, x_n]$. The *ideal quotient* $I : J$, also known as a *colon ideal*, is defined as

$$I : J = \{f \in R \,|\, \forall g \in J, fg \in I\}.$$

Ideals quotients are useful when studying differences of algebraic sets.

- Show that $I : J$ is an ideal of R.
- Show that if $J = \langle g_1, \ldots, g_s\rangle$, then

$$I : J = \bigcap_{i=1}^{s} I : \langle g_i\rangle.$$

- Show that if $g \in R$ is nonzero, then

$$I : \langle g\rangle = \frac{1}{g}(I \cap \langle g\rangle).$$

- Combine the previous observations to compute $I : J$ for the ideals $I = \langle x(x+y)^2, y\rangle$ and $J = \langle x^2, x + y\rangle$ in $\mathbb{Q}[x, y]$. You can use Problem 27 to compute intersections or you can just use the Macaulay2 method `intersect`.

Day 3

Leading terms

Recall that having fixed a monomial ordering on the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$, the largest monomial appearing with a nonzero coefficient in a polynomial f is called its *leading monomial*; we denote it $\mathrm{LM}(f)$. The coefficient of the leading monomial is called the *leading coefficient* of f; we denote it $\mathrm{LC}(f)$. The product of the two gives the *leading term* of f; we denote it $\mathrm{LT}(f)$, so we have $\mathrm{LT}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f)$.

Definition

Let I in $R = \mathbb{k}[x_1, \ldots, x_n]$ be a nonzero ideal and fix a monomial ordering on R. Denote $\mathrm{LT}(I)$ the set of leading terms of nonzero elements of I. We call $\langle \mathrm{LT}(I) \rangle$ the *ideal of leading terms* of I.

The ideal of leading terms is, by construction, a monomial ideal of R, i.e., an ideal that has a generating set consisting entirely of monomials. Although $\mathrm{LT}(I)$ is an infinite set, $\langle \mathrm{LT}(I) \rangle$ admits a finite generating set (consisting of monomials) by Hilbert's Basis Theorem. One can also show directly that a monomial ideal admits a finite generating set; this result is known as Dickson's Lemma.

One would hope that if $I = \langle f_1, \ldots, f_r \rangle$, then $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_r) \rangle$; however, this is false in general. For example, consider the polynomials $f_1 = y^2 - x$ and $f_2 = xy - 1$ in GLex. On Day 1, we showed that

$$-x^2 + y = S(f_1, f_2) \in \langle f_1, f_2 \rangle.$$

However, $x^2 \notin \langle y^2, xy \rangle$.

Theorem

Let I in $R = \mathbb{k}[x_1, \ldots, x_n]$ be a nonzero ideal and fix a monomial ordering on R. A tuple $G = (g_1, \ldots, g_s) \in R^s$ is a Gröbner basis of I if and only if $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s) \rangle$.

Since it is an equivalent characterization, this is often taken as the definition of a Gröbner basis. As it turns out, this characterization has many useful applications.

Quotient representations

We are finally able to solve our other motivational problems, namely how to represent and compare elements in the quotients of a polynomial ring.

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ and fix a monomial ordering on R. Let I in R be an ideal and let $G = (g_1, \ldots, g_s)$ be a Gröbner basis of I. Given any polynomial $f \in R$, we can use the division algorithm to write $f = \sum_{i=1}^{s} q_i g_i + r$, where r is a $\mathbb{k}$-linear combination of monomials not divisible by any of $\mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_s)$. Since G is a Gröbner basis of I, we have:

- $I = \langle g_1, \ldots, g_r \rangle$ so $f + I = r + I$;
- r is uniquely determined (it depends only on f, I, and the monomial ordering);
- no term of r is divisible by any monomial in $\langle \mathrm{LT}(I) \rangle$.

We can combine these observations into the following result.

Theorem (Macaulay's Basis Theorem)

Let I in $R = \mathbb{k}[x_1, \ldots, x_n]$ be a nonzero ideal and fix a monomial ordering on R. The monomials of R not belonging to $\langle \mathrm{LT}(I) \rangle$ form a basis of R/I as a $\mathbb{k}$-vector space. In particular, if $G = (g_1, \ldots, g_s)$ is a Gröbner basis of I, then the monomials of R not divisible by any of $\mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_s)$ form a basis of R/I as a $\mathbb{k}$-vector space.

The monomials of R not contained in $\langle \mathrm{LT}(I)\rangle$ are sometimes called the *standard monomials* modulo I.

Hilbert functions and polynomials

Recall that a polynomial $f \in R = \Bbbk[x_1, \ldots, x_n]$ is called *homogeneous of degree* d if

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for all $t \in \Bbbk \setminus \{0\}$ or, equivalently, if all terms of f have degree d. For $d \in \mathbb{N}$, denote R_d the $\Bbbk$-vector subspace of R spanned by all homogeneous polynomials of degree d, which we call the *graded component* of R of degree d. The ring R admits a direct sum decomposition

$$R = \bigoplus_{d\in\mathbb{N}} R_d$$

as a $\Bbbk$-vector space. Moreover, multiplication respects this decomposition in the sense that for all $f \in R_d, g \in R_e$ we have $fg \in R_{d+e}$. An ideal I of R is called *homogeneous* if it has a generating set consisting entirely of homogeneous polynomials. For example, monomial ideals are homogeneous. For $d \in \mathbb{N}$, let I_d be the $\Bbbk$-vector subspace of I spanned by all homogeneous polynomials of degree d in I, which we call the *graded component* of I of degree d. A homogeneous ideal I admits a direct sum decomposition

$$I = \bigoplus_{d\in\mathbb{N}} I_d$$

as a $\Bbbk$-vector space. Moreover, multiplication is compatible with this decomposition in the sense that for all $f \in I_d, g \in R_e$ we have $fg \in I_{d+e}$. When I is a homogeneous ideal, the quotient ring R/I inherits a grading

$$R/I = \bigoplus_{d\in\mathbb{N}} (R/I)_d$$

by letting $(R/I)_d$ be the span of all cosets $f + I$ with $f \in R_d$. As a $\Bbbk$-vector space, we have $(R/I)_d = R_d/I_d$. Quotients of a polynomial ring by a homogeneous ideal arise naturally as "coordinate rings" of projective varieties, so we will focus on them for the rest of this section. An analogous discussion can be had in the nonhomogeneous (i.e., affine) case.

ⓘ **Definition**

Let I be a homogeneous ideal of the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$. The *Hilbert function* of R/I is the function $H_{R/I} : \mathbb{N} \to \mathbb{N}$ defined by $H_{R/I}(d) = \dim_{\Bbbk}(R/I)_d$, i.e., the dimension of the graded component of degree d of R/I as a $\Bbbk$-vector space.

As a simple example, observe that when $I = \{0\}$ we have $R/I \cong R$ and

$$H_R(d) = \dim_{\Bbbk} R_d = \binom{n-1+d}{d}.$$

Fixing a monomial ordering on R, we have a basis of $(R/I)_d$ consisting of all monomials of degree d not contained in $\langle \mathrm{LT}(I)\rangle$. This shows the dimension of $(R/I)_d$ is always finite and gives us a practical way to compute it.

For example, consider the homogeneous ideal

$$I = \langle w^2 + x^2 + y^2 + z^2, w(x + y + z)\rangle$$

in $R = \mathbb{Q}[w, x, y, z]$ with the GRevLex ordering. We can use the following Macaulay2 code to produce the ideal of leading terms of I.

```
R=QQ[w,x,y,z]
I=ideal(w^2+x^2+y^2+z^2,w*(x+y+z))
leadTerm I
```

As a result, we get that $\langle \mathrm{LT}(I)\rangle = \langle wx, w^2, x^3\rangle$. From here, we see that

$$H_{R/I}(0) = 1, \qquad H_{R/I}(1) = 4, \qquad H_{R/I}(2) = 8$$

because all monomials of degree 0 and 1 survive in the quotient, but 2 of the 10 monomials of degree 2 are congruent to zero. For larger d, the computation is a little more involved. For example, when $d = 3$ the monomials not in $\langle wx, w^2, x^3\rangle$ are

$$wy^2, wyz, wz^2, xy^2, xyz, xz^2, x^2y, x^2z, y^3, y^2z, yz^2, z^3$$

so that $H_{R/I}(3) = 12$. In fact, for $d \geqslant 3$ the monomials not in $\langle wx, w^2, x^3\rangle$ are:

- $wy^{d-1}, wy^{d-2}z, \ldots, wyz^{d-2}, wz^{d-1}$ (d monomials),
- $xy^{d-1}, xy^{d-2}z, \ldots, xyz^{d-2}, xz^{d-1}$ (d monomials),
- $x^2y^{d-2}, x^2y^{d-3}z, \ldots, x^2yz^{d-3}, x^2z^{d-2}$ ($d-1$ monomials),
- and $y^d, y^{d-1}z, \ldots, yz^{d-1}, z^d$ ($d+1$ monomials).

Therefore, for $d \geqslant 3$ we have

$$H_{R/I}(d) = d + d + (d-1) + (d+1) = 4d.$$

We can also use Macaulay2 to compute individual values of the Hilbert function and to get bases for the graded components.

```
Q=R/I
for i to 10 do print hilbertFunction(i,Q)
basis(2,Q)
```

The behavior observed in this example generalizes.

Theorem

Let I be a homogeneous ideal in $R = \Bbbk[x_1, \ldots, x_n]$. There is a polynomial $P_{R/I}(t) \in \mathbb{Q}[t]$ such that for all d sufficiently large we have $H_{R/I}(d) = P_{R/I}(d)$.

The polynomial $P_{R/I}$ is called the *Hilbert polynomial* of R/I and it carries useful information. If the leading term of $P_{R/I}$ is ct^d, then

- $\dim(R/I) = 1 + d$, where $\dim(R/I)$ denotes the Krull dimension of R/I;
- $\deg(R/I) = cd!$, where $\deg(R/I)$ denotes the degree or multiplicity of R/I.

The dimension and the degree of R/I allow us to measure how big and complicated the vanishing locus of I is in projective space. In the example above, we have $\dim(R/I) = 2$ so the vanishing locus of I is a curve (the Krull dimension of the coordinate ring is one more than the dimension of the projective variety); also, $\deg(R/I) = 4$, so this is a curve of degree 4. To compute the Hilbert polynomial in the format above using Macaulay2 you can use the following code.

```
hilbertPolynomial(Q,Projective=>false)
```

The connection between the algebra and the geometry goes even deeper. Suppose I is a homogeneous ideal and $G = (g_1, \ldots, g_s)$ is a Gröbner basis of I. For $1 \leqslant i \leqslant s$, define polynomials

$$h_i = g_i - \mathrm{LT}(g_i)$$

obtained by removing the leading term from each g_i, and let

$$G_{i,t} = \mathrm{LT}(g_i) + th_i$$

where t is a parameter. Altogether, the polynomials $G_{i,t}$ define a family of ideals

$$I_t = \langle G_{1,t}, \ldots, G_{s,t} \rangle$$

depending on the parameter t. Note that $I_1 = I$ and $I_0 = \langle \mathrm{LT}(I) \rangle$. Our previous discussion allows us to observe that $\dim_{\Bbbk}(R/I_1)_d = \dim_{\Bbbk}(R/I_0)_d$ for all $d \in \mathbb{N}$, so that R/I_1 and R/I_0 have the same Hilbert function and, therefore, they have the same Hilbert polynomial, dimension and degree. In fact, the quotients R/I_t have the same Hilbert function for all values of the parameter t. For $t \neq 0$, the vanishing locus of I_t may look like some deformation of the vanishing locus of I. However, for $t = 0$, $I = I_0$ is a monomial ideal and its vanishing locus reduces to a union of linear subspaces; this is typically different from the vanishing locus of I but it may be easier to understand. The process of deforming the vanishing locus of I to that of I_0 is sometimes referred to as a Gröbner degeneration.

Syzygies

Finally, let us return to the division algorithm. We observed that when dividing by the terms of a Gröbner basis the remainder is unique, in particular it does not depend on the order of the divisors. However, quotients are generally not uniquely determined.

Consider the polynomials $f_1 = y^2 - x, f_2 = xy - 1, f_3 = -x^2 + y$. As we observed on Day 1, (f_1, f_2, f_3) is a Gröbner basis. We have

$$xy^2 = xf_1 - f_3 + y = (x-1)f_1 + xf_2 + (y-1)f_3 + y,$$

where the first equality was obtained using the division algorithm and y is the remainder. Thus, we have at least two different sets of coefficients $(x, 0, -1)$ and $(x-1, x, y-1)$ for f_1, f_2, f_3 that could act as "quotients" upon division of xy^2 by (f_1, f_2, f_3).

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R. Consider a tuple $F = (f_1, \ldots, f_s) \in R^s$ of nonzero elements. Given $f, r \in R$, suppose there are two different tuples $(q_1, \ldots, q_s), (\tilde{q}_1, \ldots, \tilde{q}_s) \in R^s$ such that

$$f = \sum_{i=1}^{s} q_i f_i + r = \sum_{i=1}^{s} \tilde{q}_i f_i + r.$$

Then, we have

$$\sum_{i=1}^{s} (q_i - \tilde{q}_i) f_i = 0.$$

We can study the tuples $(h_1, \ldots, h_s) \in R^s$ such that $\sum_{i=1}^{s} h_i f_i = 0$ as a way to measure the failure of uniqueness of the quotients upon division by F.

ⓘ **Definition**

Let $R = \Bbbk[x_1, \dots, x_n]$ and consider a tuple $F = (f_1, \dots, f_s) \in R^s$ of nonzero elements. A tuple $H = (h_1, \dots, h_s) \in R^s$ such that

$$\sum_{i=1}^{s} h_i f_i = 0$$

is called a *syzygy* of F. We denote $\mathrm{Syz}(F)$ the set of all syzygies of F.

The universally beloved word syzygy comes from the greek word for yoke. It is used in astronomy to describe an alignment of celestial objects. It is also the name of a few music bands and the title of several short films, TV show and podcast episodes, including an episode of the 90's cult TV show The X-Files.

The set $\mathrm{Syz}(F)$ is closed under sums and multiplication by elements of R; in other words, $\mathrm{Syz}(F)$ is a submodule of R^s. If we let

$$\mathbf{e}_i = (0, \dots, 0, \underbrace{1}_{i\text{-th position}}, 0, \dots, 0) \in R^s,$$

then we can write

$$(h_1, \dots, h_s) = \sum_{i=1}^{s} h_i \mathbf{e}_i.$$

For all choices of indices $1 \leqslant i < j \leqslant s$, we have

$$f_j \mathbf{e}_i - f_i \mathbf{e}_j \in \mathrm{Syz}(F).$$

Are there other syzygies and, if so, can we find them all? Since R is Noetherian and R^s is a finitely generated R-module, the submodule $\mathrm{Syz}(F)$ is also finitely generated. Thus, to describe all syzygies it is enough to find a finite generating set.

Going back to our example, we know that f_3 is the S-polynomial of f_1 and f_2:

$$S(f_1, f_2) = \frac{xy^2}{y^2}(y^2 - x) - \frac{xy^2}{xy}(xy - 1) = -x^2 + y = f_3,$$

where xy^2 is the least common multiple of the leading monomials of f_1 and f_2. We know that the S-polynomial is designed to cancel the leading terms of its arguments, a fact which we can write as follows.

$$S(\mathrm{LT}(f_1), \mathrm{LT}(f_2)) = \frac{xy^2}{y^2}y^2 - \frac{xy^2}{xy}xy = x(y^2) - y(xy) = 0$$

If we let $\mathrm{LT}(F) = (\mathrm{LT}(f_1), \mathrm{LT}(f_2), \mathrm{LT}(f_3))$, the above equality can be reintepreted using the language of syzygies: $(x, -y, 0) \in \mathrm{Syz}(\mathrm{LT}(F))$.

As observed in our example, S-polynomials give rise to syzygies of leading terms. In fact, every syzygy among leading terms arises as an R-linear combination of these.

Theorem

Let $R = \Bbbk[x_1, \ldots, x_n]$ and consider a tuple $F = (f_1, \ldots, f_s) \in R^s$ of nonzero elements. Fix a monomial ordering on R and write $\mathrm{LT}(F)$ for the tuple $(\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_s)) \in R^s$. The elements

$$\sigma_{ij} = \frac{\mathrm{lcm}(\mathrm{LM}(f_i), \mathrm{LM}(f_j))}{\mathrm{LT}(f_i)}\mathbf{e}_i - \frac{\mathrm{lcm}(\mathrm{LM}(f_i), \mathrm{LM}(f_j))}{\mathrm{LT}(f_j)}\mathbf{e}_j$$

for $1 \leqslant i < j \leqslant s$ generate the submodule $\mathrm{Syz}(\mathrm{LT}(F))$ of R^s.

In our ongoing example, we have:

$$\sigma_{12} = (x, -y, 0), \qquad \sigma_{13} = (x^2, 0, y^2), \qquad \sigma_{23} = (0, x, y).$$

Notice that $\sigma_{13} = x\sigma_{12} + y\sigma_{23}$, so σ_{13} is a redundant generator. We can also see that σ_{13} is related to one of the "obvious" syzygies of F:

$$\tau_{13} = -f_3\mathbf{e}_1 + f_1\mathbf{e}_3 = (x^2 - y, 0, y^2 - x) = \sigma_{13} - (y, 0, x).$$

The tuple $(y, 0, x)$ happens to contain the quotients of division of $S(f_1, f_3)$ upon division by F:

$$S(f_1, f_3) = -x^3 + y^3 = y \cdot f_1 + 0 \cdot f_2 + x \cdot f_3.$$

In this case, we say that σ_{13} "lifts" to a syzygy of F. Replicating these steps with σ_{12} and σ_{23}, we get the quotient tuples

$$\begin{aligned} S(f_1, f_2) &= -x^2 + y = 0 \cdot f_1 + 0 \cdot f_2 + 1 \cdot f_3 \leftrightsquigarrow (0, 0, 1), \\ S(f_2, f_3) &= y^2 - x = 1 \cdot f_1 + 0 \cdot f_2 + 0 \cdot f_3 \quad \leftrightsquigarrow (1, 0, 0), \end{aligned}$$

so σ_{12} and σ_{23} lift to the following syzygies of F:

$$\begin{aligned} \tau_{12} &= (x, -y, 0) - (0, 0, 1) = (x, -y, -1), \\ \tau_{23} &= (0, x, y) - (1, 0, 0) = (-1, x, y). \end{aligned}$$

Here is the crucial observation: in order to write every S-polynomial $S(f_i, f_j)$ as a linear combination of F we want the remainder of $S(f_i, f_j)$ upon division by F to be zero; in other words, we want F to be a Gröbner basis!

Theorem

Let $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial ordering on R. A tuple $G = (g_1, \ldots, g_s) \in R^s$ of nonzero elements is a Gröbner basis if and only if every homogeneous element of $\mathrm{Syz}(\mathrm{LT}(G))$ lifts to an element of $\mathrm{Syz}(G)$. In this case, if $\sigma_1, \ldots, \sigma_m$ are homogeneous elements generating $\mathrm{Syz}(\mathrm{LT}(G))$, then their lifts $\tau_1, \ldots, \tau_m$ generate $\mathrm{Syz}(G)$.

We can formalize the process for finding generators of $\mathrm{Syz}(G)$ in the following algorithm.

Lifting syzygies

To find a generating set of $\mathrm{Syz}(G)$ where $G = (g_1, \dots, g_s)$ is a Gröbner basis, proceed as follows.

1. For all indices $1 \leqslant i < j \leqslant s$, compute
$$\sigma_{ij} = \frac{\operatorname{lcm}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))}{\mathrm{LT}(g_i)}\mathbf{e}_i - \frac{\operatorname{lcm}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))}{\mathrm{LT}(g_j)}\mathbf{e}_j.$$
2. For all indices $1 \leqslant i < j \leqslant s$, find $(c_{ij1}, \dots, c_{ijs}) \in R^s$ such that
$$S(g_i, g_j) = \sum_{k=1}^{s} c_{ijk} g_k.$$
3. For all indices $1 \leqslant i < j \leqslant s$, compute
$$\tau_{ij} = \sigma_{ij} - \sum_{k=1}^{s} c_{ijk} \mathbf{e}_k.$$
4. Return the set $\{\tau_{ij} \,|\, 1 \leqslant i < j \leqslant s\}$.

Macaulay2 can find syzygies using the command `syz`.

```
R=QQ[x,y,MonomialOrder=>GLex]
I=ideal(y^2-x,x*y-1)
--syzygies of the leading terms
LTG=leadTerm I
syz LTG
--syzygies of the Gröbner basis
G=gens gb I
syz G
```

Day 3 problems

Problem 29

Consider the ideal in Problem 12.

- Find the initial ideal with respect to GRevLex.
- Find the initial ideal with respect to Lex.

Problem 30

Consider the ideal of $R = \mathbb{Q}[x, y, z]$ generated by the equations in Problem 17. Fix a monomial ordering on R.

- Find a basis of R/I as a $\mathbb{k}$-vector space and show it is finite dimensional.
- If you previously solved Problem 17, how does the dimension of R/I relate to the total number of solutions of the system?

Problem 31

Let $R = \mathbb{C}[x, y, z]$ and

$$I = \langle y^2z - yz^2, xyz, x^2z - xz^2, x^2y - xy^2 \rangle.$$

- Verify that the generators of I form a Gröbner basis with respect to GRevLex.
- Use the initial ideal of I to find the Hilbert polynomial of R/I.
- Find the dimension and degree of R/I, then use them to give a geometric description of $\mathbb{V}(I)$ in $\mathbb{P}^2$.

Problem 32

Let $R = \Bbbk[w, x, y, z]$ and let J be the defining ideal of the twisted cubic in $\mathbb{P}^3$ that you constructed in Problem 21.

- Find a Gröbner basis G of J with respect to GRevLex.
- Use G to find the initial ideal of J and, from there, the Hilbert polynomial of R/J.
- Find the dimension of R/I to confirm that $\mathbb{V}(I)$ is a curve (remember the dimension of the ring is the dimension of the projective variety plus one).
- Find the degree of R/I to confirm that $\mathbb{V}(I)$ is a cubic.
- Show that $\operatorname{Syz}(G)$ is generated by two linear syzygies.

Problem 33

This problem is about studying a non-homogeneous ideal by making it homogeneous. We start with a brief review of projective space.

The projective space $\mathbb{P}^n$ over the field $\Bbbk$ is made up of points $[x_0 : x_1 : \cdots : x_n]$ with at least one $x_i \neq 0$, and these points are defined up to nonzero scalars, meaning that for every $0 \neq \lambda \in \Bbbk$ we have

$$[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n].$$

A homogeneous polynomial f of degree d has the property that

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n)$$

so the vanishing of a homogeneous polynomial at a point of $\mathbb{P}^n$ is well-defined. The affine space $\mathbb{A}^n$ over $\Bbbk$ embeds embeds in $\mathbb{P}^n$ by sending $(x_1, \ldots, x_n)$ to $[1 : x_1 : \cdots : x_n]$. Now, if $X \subseteq \mathbb{A}^n$ is a variety, the smallest subvariety of $\mathbb{P}^n$ that contains the image of X under this embedding is called the *projective closure* of X.

If $X \subseteq \mathbb{A}^n$ is the vanishing locus of an ideal $I = \langle f_1, \ldots, f_r \rangle \subseteq \Bbbk[x_1, \ldots, x_n]$, the first thing you might try to do is multiply the terms in the generators f_i by powers of the variable x_0 so that the resulting polynomials f_i^h are homogeneous (see the example below). The following exercises show that this approach can fail even in simple situations.

- Let $I = \langle f_1, f_2 \rangle \subseteq \mathbb{C}[x, y]$ where $f_1 = y^2 - x$ and $f_2 = xy - 1$. Show that the vanishing locus of I in $\mathbb{A}^2$ contains exactly three points. We can denote these points (a_i, b_i) for $i \in \{1, 2, 3\}$.
- When we homogenize f_1 and f_2 with respect to a new variable z, we get $f_1^h = y^2 - xz$ and $f_2^h = xy - z^2$. Show that the vanishing locus of the homogeneous ideal $\langle f_1^h, f_2^h \rangle \subseteq \mathbb{C}[x, y, z]$ in $\mathbb{P}^2$ contains $[a_i : b_i : 1]$ for $i \in \{1, 2, 3\}$, and one additional point $[a_4 : b_4 : 0]$.

Here is how we remedy the situation.

- Compute a Gröbner basis G of I with respect to a degree compatible monomial ordering such as GLex or GRevLex.
- If $G = (g_1, \dots, g_s)$, then we form the ideal $I^h = \langle g_1^h, \dots, g_s^h \rangle$ of $\mathbb{C}[x, y, z]$ generated by the homogenizations of the elements in G with respect to z.
- Show that the vanishing locus of I^h in $\mathbb{P}^2$ contains only the points $[a_i : b_i : 1]$ for $i \in \{1, 2, 3\}$.

The general theory says that if $G = \langle g_1, \dots, g_s \rangle$ is a Gröbner basis of an ideal $I \subseteq \Bbbk[x_1, \dots, x_n]$ with respect to a degree compatible monomial ordering, then the projective closure of $\mathbb{V}(I)$ in $\mathbb{P}^n$ is the vanishing locus of $I^h = \langle g_1^h, \dots, g_s^h \rangle \subseteq \Bbbk[x_0, x_1, \dots, x_n]$.

Problem 34

This is a continuation of Problem 19 on Shidoku puzzles. Let $I \subseteq \mathbb{Q}[a, \dots, p]$ be the ideal representing all possible Shidoku boards.

- Find I^h using the method described in Problem 31. Macaulay2 has the `homogenize` method that you can use to homogenize a polynomial or an ideal with respect to a variable (you will need to work in a larger polynomial ring that contains one extra variable).
- Show that R/I^h has dimension one. This tells you that the projective variety defined by I^h has dimension zero or, equivalently, that it is a finite set of points (whose coordinates are the entries of all the possible Shidoku boards).
- Find the degree of R/I^h. This will tell you the number of points in the vanishing locus of I^h, which is also the total number of possible Shidoku boards.
- Use similar methods to find the number of possible solutions for the following board.

			4
		2	
	3		
1			

Problem 35

Let $R = \mathbb{Q}[w, x, y, z]$. The tuple

$$G = (x^2 + yz, wx + yz, w^2 + yz, wyz - xyz) \in R^4$$

is a Gröbner basis with respect to GRevLex.

- Construct all the generators σ_{ij} of $\operatorname{Syz}(\operatorname{LT}(G))$.
- Remove all non-minimal σ_{ij} (this will reduce computations in the next steps).
- Find lifts τ_{ij} of the minimal σ_{ij} to construct a generating set of $\operatorname{Syz}(G)$.

Problem 36

This problem tries to clarify what it means to "lift" a syzygy.

Let $R = \Bbbk[x_1, \dots, x_n]$ and consider a tuple $G = (g_1, \dots, g_s) \in R^s$ of nonzero elements. Fix a monomial ordering on R and write $\operatorname{LT}(G)$ for the tuple $(\operatorname{LT}(g_1), \dots, \operatorname{LT}(g_s)) \in R^s$. A tuple of terms $(t_1, \dots, t_s) \in R^s$ is *homogeneous* of degree $a \in \mathbb{N}^n$ relative to G if $\operatorname{LM}(g_i) \operatorname{LM}(t_i) = x^a$ for all $i \in \{1, \dots, s\}$ such that $t_i \neq 0$.

- Show that for $1 \leqslant i < j \leqslant s$ the element

  $$\sigma_{ij} = \frac{\operatorname{lcm}(\operatorname{LM}(g_i), \operatorname{LM}(g_j))}{\operatorname{LT}(g_i)} \mathbf{e}_i - \frac{\operatorname{lcm}(\operatorname{LM}(g_i), \operatorname{LM}(g_j))}{\operatorname{LT}(g_j)} \mathbf{e}_j \in R^s$$

  is homogeneous of degree a relative to G where $x^a = \operatorname{lcm}(\operatorname{LM}(g_i), \operatorname{LM}(g_j))$.

Every element of R^s decomposes as a sum of homogeneous elements of possibly different degrees relative to G.

- For example, consider the tuple

  $$G = (y^2 - x, xy - 1, -x^2 + y)$$

  of polynomials in $\mathbb{Q}[x, y]$ with GLex. Decompose

  $$H = (x^3y - xy^2, x^3 + y^3, xy^3 - x^2y)$$

  into a sum of homogeneous elements relative to G.

Given $H \in R^s$, we write $H = \sum_{a \in \mathbb{N}^n} H_a$ with H_a homogeneous of degree a relative to G. We define the *leading form* of H relative to G as $\operatorname{LF}_G(H) = H_d$ where

$$x^d = \max\{x^a \mid H_a \neq 0\}$$

taken with respect to the monomial ordering.

- Find $\operatorname{LF}_G(H)$ for the triples G and H above.
- In general, show that if $H \in \operatorname{Syz}(G)$, then $LF_G(H) \in \operatorname{Syz}(\operatorname{LT}(G))$.

Thus, the operator LF_G defines a function from R^s to R^s that sends the submodule $\operatorname{Syz}(G)$ to $\operatorname{Syz}(\operatorname{LT}(G))$. Finally, we say that $H \in R^s$ is a *lifting* of $\overline{H} \in R^s$ if $\operatorname{LF}_G(H) = \overline{H}$.

- For the triple G above, find a lifting of

  $$\overline{H} = (x^4y^2, x^3y^3, x^2y^4).$$

Not every element $\overline{H} \in R^s$ has a lifting. However, if G is a Gröbner basis and $\overline{H}$ is homogeneous, then $\overline{H}$ has a lifting.

4. Characteristic p methods (J. Jeffries)

This course introduced key positive characteristic methods, including a suite of techniques used to study problems in commutative algebra and algebraic geometry that makes use of the Frobenius morphism. This course was taught by Jack Jeffries (University of Nebraska).

4.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

4.2 Lecture Notes and Tutorials

We have included copies of Jack's lecture notes and his tutorials, which were provided by Jack. The tutorials are also found in this document.

THE FROBENIUS MAP: THE POWER OF PRIME CHARACTERISTIC

JACK JEFFRIES

These are lecture notes and exercises for a short graduate lecture series on positive characteristic methods for the SLMath/SMS Summer School An Introduction to Recent Trends in Commutative Algebra in June 2025. My goal in this series is to give an appreciation for the power of techniques involving the Frobenius map to prove statements that have nothing to do with Frobenius. It is not my goal to thoroughly develop the tools needed for research in this area. The audience has a varied background, so I am not assuming any background beyond a first year graduate sequence on algebra. There is not enough time in this course to cover background material from commutative algebra and homological algebra in addition to the specific content of these lectures, so instead I will often give statements that are specialized to more concrete situations rather than giving the most general statements, and sometimes also offer a "more generally version" for those have have additional background. For time reasons, I will often sketch proofs, occasionally leaving some details to the exercises.

In the first lecture, I will discuss the basic perspectives and terminology of the Frobenius map. The first problem set is intended to solidify these notions, though there are also a few problems that build towards the later lectures. The second lecture will briefly introduce tight closure and an application. The third lecture will introduce a couple of notions of F-singularities and outline a couple more applications. The second problem set will explore the notions from the last two lectures, and fill in some details of the proofs.

Throughout these notes, all rings are commutative with $1 \neq 0$, and p will denote a positive prime integer.

1. Basics with the Frobenius map

Recall that a ring R has characteristic p if

$$p = \underbrace{1 + \cdots + 1}_{p \text{ times}}$$

is zero in R. This is equivalent to R containing a field of characteristic p as a subring: if R has characteristic p, the image of the homomorphism $\mathbb{Z} \longrightarrow R$ is isomorphic to $\mathbb{F}_p$.

The Frobenius map. Let us start with an observation about binomial coefficients. For any integer i with $0 < i < p$, the binomial coefficient

$$\binom{p}{i} = \frac{p!}{(p-i)! \cdot i!}$$

has a factor of p in the numerator, but not the denominator. Since we also know this coefficient is an integer, e.g., for combinatorial reasons, the Fundamental Theorem of Arithmetic says that it is a multiple of p. Thus, when R has characteristic p, for any $r, s \in R$, one has

$$\begin{aligned}(r+s)^p &= r^p + \binom{p}{1} r^{p-1}s + \binom{p}{2} r^{p-2}s^2 + \cdots + \binom{p}{p-1} rs^{p-1} + s^p \\ &= r^p + s^p, \quad \text{and} \\ (rs)^p &= r^p s^p,\end{aligned}$$

and $1^p = 1$, so the map

$$F\colon R \longrightarrow R, \quad F(r) = r^p$$

is a ring homomorphism from R to itself, called the **Frobenius map** on R. We may denote this as F_R to indicate the ring when useful.

One can apply the Frobenius map multiple times:

$$F^e\colon R \longrightarrow R, \quad F^e(r) = r^{p^e}$$

which we may call the **e-th Frobenius** or **e-th Frobenius iterate**. Note that no power map is a ring homomorphism in characteristic zero.

Example 1.1. For $R = \mathbb{F}_p$ the Frobenius map is the identity: this is Fermat's Little Theorem.

Example 1.2. For $R = \mathbb{F}_p[x]$, the Frobenius map is given by

$$F(a_n x^n + \cdots + a_1 x + a_0) = a_n x^{pn} + \cdots + a_1 x^p + a_0$$

and the iterates by

$$F^e(a_n x^n + \cdots + a_1 x + a_0) = a_n x^{p^e n} + \cdots + a_1 x^{p^e} + a_0.$$

Every ring of characteristic p has a Frobenius map, and the Frobenius map is compatible with every ring homomorphism between rings of characteristic p:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "F_R"'] & S \arrow[d, "F_S"] & r \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(r) \arrow[d, mapsto] \\ R \arrow[r, "\varphi"] & S & r^p \arrow[r, mapsto] & \varphi(r^p) = \varphi(r)^p. \end{tikzcd}$$

This universality and naturality is a clear sign of the importance of the Frobenius map.

Injectivity and surjectivity. Let us start with a simple relationship between the Frobenius map and something that has nothing to do with it.

Lemma 1.3. *Let R be a ring of characteristic p. The Frobenius map on R is injective if and only if R is reduced (meaning that R has no nonzero nilpotents).*

Proof. We will prove the contrapositive of each direction. $(\Leftarrow)$: If F_R is not injective, then there is some $r \neq 0$ with $r^p = 0$; such an element is a nonzero nilpotent of R.

$(\Rightarrow)$: If R is not reduced, then there is some $r \neq 0$ with $r^n = 0$ for some $n \geq 2$. Take n maximal such that $r^n \neq 0$; then $np > n$, so $F(r^n) = r^{pn} = 0$, and r^n is a nonzero element of the kernel of F_R. ☐

It is rarer for the Frobenius map to be surjective. The image of the Frobenius map is evidently the p-th powers of elements in R. A ring of positive characteristic is **perfect** if its Frobenius map is bijective. You are likely familiar with this consideration for fields. Perfect fields include all finite fields, like $\mathbb{F}_p$ and $\mathbb{F}_{p^7}$, and all algebraically closed fields, like $\overline{\mathbb{F}_p}$ and $\overline{\mathbb{F}_p(t)}$. However, a field like $\mathbb{F}_p(t)$ is not perfect, as t is not a p-th power. However $\mathbb{F}_p[x]$ is evidently not perfect. One can show that when R is Noetherian then F_R is surjective if and only if R is a finite product of perfect fields.

Alternative perspectives. One of the most confusing aspects of the Frobenius map is the fact that the source and target are the same, though the map is typically not an isomorphism. It is often useful to separate the source and target of the Frobenius to clarify the situation. One can think of this as analogous to the case of linear algebra, where some aspects of an endomorphism of a vector space are easier to understand with separate bases on the source and target.

Our first alternative perspective on Frobenius is based on renaming the target copy of R. We will decorate every element in the target of the e-th Frobenius F^e with the decoration F^e_*. That is, $F^e_* R$ is just an collection of doppelgängers of elements R:

$$\begin{aligned} &F^e_* R = \{F^e_* r \mid r \in R\} \\ &F^e_* r + F^e_* s = F^e_*(r+s) \quad \text{and} \quad F^e_* r F^e_* s = F^e_*(rs), \end{aligned}$$

so the map

$$R \longrightarrow F^e_* R \qquad r \longmapsto F^e_* r$$

is an isomorphism. After rewriting "target R" as $F^e_* R$ via the isomorphism above, the e-th Frobenius map takes the form

$$R \longrightarrow F^e_* R \qquad r \longmapsto F^e_*(r^{p^e}).$$

One should think of this as follows: the e-th Frobenius map sends $r \longrightarrow r^{p^e}$, and the F^e_* symbol simply says which copy of R the element r^{p^e} lives in. Put another way, we have the commutative diagram

$$\begin{tikzcd} R \arrow[r, "F^e"] \arrow[d, "="'] & R \arrow[d, "\cong"] & & r \arrow[r, mapsto] \arrow[d, mapsto] & r^{p^e} \arrow[d, mapsto] \\ R \arrow[r] & F^e_* R & & r \arrow[r, mapsto] & F^e_*(r^{p^e}) \end{tikzcd}$$

where the bottom row is the Frobenius from $R \longrightarrow F^e_* R$ and the right map is the isomorphism "adding the decoration F^e_*".

When R is a domain, there is another useful way to think of $F^e_* R$. In this case, R has a field of fractions K, which admits an algebraic closure $\overline{K}$. Every element of R has a unique p^e-th root r^{1/p^e} in $\overline{K}$, as $\overline{K}$ is a perfect field. Define

$$R^{1/p^e} := \{r^{1/p^e} \in \overline{K} \mid r \in R\}.$$

One can verify that R^{1/p^e} is a subring of $\overline{K}$, and the map

$$R \longrightarrow R^{1/p^e} \qquad r \longmapsto r^{1/p^e}$$

is a ring isomorphism. We can think of the exponent ${}^{1/p^e}$ as a decoration that yields an isomorphic copy of R. After rewriting "target R" as R^{1/p^e} via this isomorphism, the Frobenius map takes the form

$$R \longrightarrow R^{1/p^e} \qquad r \longmapsto (r^{p^e})^{1/p^e} = r.$$

That is, after the identification above, the Frobenius map identifies with the inclusion of $R \subseteq R^{1/p^e}$. Put another way, we have the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "F^e"] \arrow[d, "="'] & R \arrow[d, "\cong"] & r \arrow[r, mapsto] \arrow[d, mapsto] & r^{p^e} \arrow[d, mapsto] \\
R \arrow[r] & R^{1/p^e} & r \arrow[r, mapsto] & r = (r^{p^e})^{1/p^e},
\end{tikzcd}$$

where the bottom row is the inclusion map and the right map is the isomorphism $R \cong R^{1/p^e}$ of taking p^e-th roots. This notion of roots equally well makes sense when R is reduced: in this case, R embeds into product of fields, which embeds into a product of algebraically closed fields, where every element again has a unique p^e-th root.

A third perspective on the Frobenius on a reduced ring is by identifying the source of Frobenius with R^{p^e}, the subring consisting of p^e-th powers of elements of R. In this case, the Frobenius map corresponds to the inclusion map $R^{p^e} \subseteq R$.

Typical constructions. We now discuss some typical constructions for ring maps applied to special case of the Frobenius. For a general ring homomorphism $\varphi : A \longrightarrow B$, one has the notion of extension of an ideal $I \subseteq A$ given as the ideal of B given by $(\varphi(a) \mid a \in I)$. This leads to the notion of Frobenius powers. Given an ideal $I \subseteq R$, we define the **Frobenius powers** of I as

$$I^{[p^e]} = (a^{p^e} \mid a \in I) = (F^e(a) \mid a \in I).$$

If $I = (a_1, \dots, a_t)$, then $I^{[p^e]} = (a_1^{p^e}, \dots, a_t^{p^e})$, as is the case in general for extension of ideals. Observe that $I^{[p^e]} \subseteq I^{p^e}$, but these are typically different when I is not principal.

Another important construction comes from restriction of scalars. For a general ring homomorphism $\varphi : A \longrightarrow B$, one can view B as an A-module by restriction of scalars: B becomes an A-module by the rule $a \cdot b = \varphi(a)b$. One can view R as an R-module by restriction of scalars through F^e, so R acts on R by the rule

$$r \cdot s = r^{p^e} s.$$

It is especially helpful to use the alternative notations for the Frobenius map in this setting. Consider the Frobenius map in the form

$$R \longrightarrow F_*^e R \quad r \longmapsto F_*^e(r^{p^e}).$$

The R-module action on $F_*^e R$ is then

$$r \cdot F_*^e s = F_*^e(r^{p^e} s).$$

For R reduced, we may also consider the Frobenius map in the form

$$R \subseteq R^{1/p^e}.$$

The R-module action on R^{1/p^e} is then the straightforward action

$$r \cdot s^{1/p^e} = rs^{1/p^e} = (r^{p^e} s)^{1/p^e}.$$

We will return to discuss this structure in great detail for a polynomial ring soon.

One can also apply the restriction of scalars to an arbitrary R-module. For a general ring homomorphism $\varphi : A \longrightarrow B$, and B-module N, one can view N as an A-module by restriction of scalars: N becomes an A-module by the rule $a \cdot n = \varphi(a)n$. To apply this with the Frobenius map, we let M be an R-module. Let us think of the Frobenius map in the form

$$R \longrightarrow F_*^e R \quad r \longmapsto F_*^e(r^{p^e}),$$

and think of M as a module over the target; we will rewrite M as

$$F_*^e M = \{F_*^e m \mid m \in M\}$$

with $F_*^e R$-action

$$F_*^e r \cdot F_*^e m = F_*^e(rm).$$

The action of R on $F_*^e M$ is then

$$r \cdot F_*^e m = F_*^e(r^{p^e}) F_*^e m = F_*^e(r^{p^e} m).$$

Finally, we discuss extension of scalars. For a general ring homomorphism $\varphi : A \longrightarrow B$, and A-module M, one can create a new B-module by extension of scalars. The construction is most naturally stated in terms of tensor products, but we give a slightly more concrete construction. One can write M in terms of generators and relations: M has generating set $\{m_i\}_i$ with relations $\{\sum_i a_{ij} m_i\}_j$, meaning $\sum_i a_{ij} m_i = 0$ in M for all j, and that these generate the tuples of relations on these generators. The module $\varphi^* M$ is then the B-module with generating set $\{m_i\}_i$ with relations $\{\sum_i \varphi(a_{ij}) m_i\}_j$. To apply this with the Frobenius map, we let M be an R-module. If M is as above, the Frobenius restriction of scalars module is the R-module $F^{e*}(M)$ with generating set $\{m_i\}_i$ with relations $\{\sum_i a_{ij}^{p^e} m_i\}_j$.

Polynomial rings and Kunz' Theorem. We will now analyze the R-module structure of $F_*^e R$ in detail in an important case.

Theorem 1.4. *Let K be a perfect field of characteristic p, and $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over K. Then $F_*^e S$ is a free S-module with basis*

$$B = \{F_*^e(x_1^{a_1} \cdots x_n^{a_n}) \mid 0 \le a_i < p^e\}.$$

Proof. We need to show that every element of $F_*^e S$ can be written as an S-linear combination of the elements above.

Every element of $F_*^e S$ is a sum of elements of the form $F_*^e(\gamma x_1^{b_1} \cdots x_n^{b_n})$ with $\gamma \in K$ and $b_1, \dots, b_n \ge 0$. Write $b_i = p^e c_i + a_i$ with $0 \le a_i < p^e$. Then

$$\begin{aligned} F_*^e(\gamma x_1^{b_1} \cdots x_n^{b_n}) &= F_*^e(\gamma x_1^{p^e c_1 + a_1} \cdots x_n^{p^e c_n + a_n}) \\ &= F_*^e(\gamma x_1^{p^e c_1} \cdots x_n^{p^e c_n}) F_*^e(x_1^{a_1} \cdots x_n^{a_n}). \\ &= \gamma^{1/p^e} x_1^{c_1} \cdots x_n^{c_n} \cdot F_*^e(x_1^{a_1} \cdots x_n^{a_n}) \end{aligned}$$

Note that we have used that K is perfect in the last step. This shows that the purported basis spans.

To see this set is linearly independent, suppose that we have some $\beta_1, \dots, \beta_t \in B$ and $s_1, \dots, s_t \in S$ such that $\sum_i s_i \beta_i = 0$. Note that in a product

$$s_i \beta_i = s_i \cdot F_*^e(x_1^{a_1} \cdots x_n^{a_n}) = F_*^e(s_i^{p^e} x_1^{a_1} \cdots x_n^{a_n}),$$

every monomial occurring in the polynomial $s_i^{p^e} x_1^{a_1} \cdots x_n^{a_n}$ has exponents $b_1, \dots, b_n$ such that $b_i \equiv a_i \mod p^e$. In particular, writing each $s_i \beta_i$ as F_*^e of some polynomial as above, the polynomials that occur have mutually distinct monomials, and thus cannot cancel each other.

It follows that $s_i\beta_i = 0$ for each i, which implies $s_i = 0$ for each i. This shows that B is a free basis. ☐

Intuitively, this proof shows that viewing S as the S-module $F^e_* S$ breaks apart into pieces of the form $S \cdot F^e_*(x_1^{a_1} \cdots x_n^{a_n})$ consisting of all polynomials whose exponent vectors are coordinatewise congruent to $(a_1, \dots, a_n)$. Various applications of the Frobenius are based on taking an element of S, viewing it as an element $F^e_* S$, and breaking it into its components in this free S-basis, or equivalently, applying S-linear maps from $F^e_* S$ back to S. We will return to this idea soon.

This decomposition a special case of the "Fundamental Theorem of Frobenius".

Theorem 1.5 (Kunz). *Let R be a Noetherian ring of characteristic p, and let $e \geq 1$. The module $F^e_* R$ is a flat R-module if and only if R is a regular ring.*

A flat module is a weakening of free module (free implies flat), and a polynomial ring over a field is a key example of a regular ring.

We end with a technical definition that is useful for many purposes.

Definition 1.6. A ring R of characteristic p is **F-finite** if $F_* R$ is a finitely generated R-module; equivalently, $F^e_* R$ is a finitely generated R-module for all e.

This is a finiteness property, somewhat akin to Noetherianity. In the exercises, you will show that every finitely generated algebra over a perfect field is F-finite. We can get a more concrete version of Kunz' theorem when R is F-finite and local. Recall that a **local ring** is a ring with a unique maximal ideal. We often write $(R, \mathfrak{m})$ for a local ring to denote R and its maximal ideal, or $(R, \mathfrak{m}, k)$ to denote the residue field $k = R/\mathfrak{m}$ as well. Given any ring R and prime ideal $\mathfrak{p}$, we can obtain a local ring $R_{\mathfrak{p}}$ for adjoining inverses to every element outside of $\mathfrak{p}$, a process called localization.

A typical example of a local ring is, for a field K and some variables $x_1, \dots, x_n$, the collection of rational functions for the form

$$\left\{ \frac{f(x)}{g(x)} \mid g(x) \text{ has nonzero constant term} \right\}.$$

This is the local ring $K[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ obtained from the polynomial ring by localization at the prime (maximal) ideal consisting of polynomials with constant term zero. Another key example of a local ring is the power series ring $K[\![x_1, \dots, x_n]\!]$. These are the two typical examples to keep in mind of regular local rings.

Corollary 1.7 (Kunz). *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring of characteristic p. The module $F^e_* R$ is a free R-module if and only if R is a regular ring.*

Example 1.8. If K is a perfect field and S is either

$$K[x_1, \dots, x_n]_{(x_1, \dots, x_n)} \quad \text{or} \quad K[\![x_1, \dots, x_n]\!],$$

then $F^e_* S$ is free with basis

$$B = \{F^e_*(x_1^{a_1} \cdots x_n^{a_n}) \mid 0 \leq a_i < p^e\}$$

as in the polynomial case.

Exercise set #1

Throughout this problem set all rings have characteristic p.

(1) * Convince yourself, as succinctly as possible, that $r \in I$ if and only if $F^e_* r \in F^e_* I$.
(2) Let $S = \mathbb{F}_3[x,y]$. Find an element in $(x,y)^3$ that is not in $(x,y)^{[3]}$.
(3) Let $S = \mathbb{F}_3[x,y]$. Write out the free basis B for $F_* S$ from the proof of Theorem 1.4 and write the element $F_*(2x^6y^7 + x^5y^3 + x^3y^4 + 2xy^2)$ as an S-linear combination of B.
(4) Let $\mathfrak{p}$ be a prime ideal in R. Show that $F^{-1}(\mathfrak{p}) = \mathfrak{p}$.
(5) * Let R be a ring and I be an ideal. Show that $F^e_*(I^{[p^e]}) = I F^e_*(R)$.
(6) Show that $R^{p^e} = \{r^{p^e} \mid r \in R\}$ is a subring of R.
(7) Suppose that R is reduced. Show that $R \cong R^{p^e}$, and that after identifying the source of the e-th Frobenius map with R^{p^e} via the isomorphism you found, the Frobenius map identifies with the inclusion map $R^{p^e} \subseteq R$.
(8) * Let $R = \mathbb{F}_p[x,y]/(xy)$.
   (a) Explain why R has $\mathbb{F}_p$-vector space basis $\{1, x, x^2, x^3, \dots, y, y^2, y^3, \dots\}$ (where, by abuse of notation, we write x for the equivalence class of x in the quotient).
   (b) Find an $\mathbb{F}_p$-vector space basis for $F^e_* R$, and describe the action of R on $F^e_* R$ explicitly in terms of the action of each basis element of R with each basis element of $F^e_* R$.
   (c) Show that the ideal (x) of multiples of x in R is isomorphic to $R/(y)$ as an R-module.
   (d) Show that, as R-modules,

$$F^e_* R \cong R \cdot F^e_* 1 \oplus \bigoplus_{0<i<p^e} R/(y) \cdot F^e_*(x^i) \oplus \bigoplus_{0<j<p^e} R/(x) \cdot F^e_*(y^j).$$

(9) Let $R = \mathbb{F}_2[x^2, xy, y^2]$; i.e., R is the subring of the polynomial ring $\mathbb{F}_2[x,y]$ with $\mathbb{F}_2$ vector space basis consisting of $\{x^i y^j \mid i+j \text{ is even}\}$. Find a generating set for $F_* R$ as an R-module. Is your generating set a free basis?
(10) Let $K = \mathbb{F}_p(t_1, t_2, t_3, \dots)$, the field of rational functions over $\mathbb{F}_p$ in countably many variables. Is K an F-finite field?
(11) (a) Let R be an F-finite ring and I be an ideal. Show that R/I is also F-finite.
   (b) Let R be an F-finite ring and x be an indeterminate. Show that $R[x]$ is also F-finite. Deduce that every finitely generated algebra over a perfect field is F-finite.
(12) Let R be as in (9). Verify directly that $F_* R$ has no free basis.
   It may be useful to use the fact that if M is a free R-module with basis B and I is an ideal, then M/IM is a free R/I-module with basis given by the images of B; try different maximal ideals.

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*To be used later in the lectures.

(13) [§] Let K be a perfect field and $S = K[x_1,\dots,x_n]$. Consider $\mathrm{Hom}_S(F^e_* S, S)$, the set of S-linear maps from $F^e_* S$ to S. Let $A = \{(a_1,\dots,a_n) \in \mathbb{Z}^n \mid 0 \le a_i < p^e\}$.
  (a) Show that for each $\alpha \in A$, there is a map $\Phi_\alpha \in \mathrm{Hom}_S(F^e_* S, S)$ such that

$$\Phi_\alpha(F^e_*(x_1^{a_1}\cdots x_n^{a_n})) = \begin{cases} 1 & \text{if } (a_1,\dots,a_n) = \alpha \\ 0 & \text{if } (a_1,\dots,a_n) \in A \setminus \{\alpha\}. \end{cases}$$

  (b) Consider $\mathrm{Hom}_S(F^e_* S, S)$ as an S-module by the rule $s \cdot \varphi(-) = s\varphi(-)$. Show that $\mathrm{Hom}_S(F^e_* S, S)$ is a free S-module with this action, and find a basis.
  (c) Consider $\mathrm{Hom}_S(F^e_* S, S)$ as an $F^e_* S$-module by the rule $F^e_* s \cdot \varphi(-) = \varphi(F^e_* s \cdot -)$. Show that $\mathrm{Hom}_S(F^e_* S, S)$ is a free $F^e_* S$-module with basis the singleton $\{\Phi := \Phi_{(p^e-1,\dots,p^e-1)}\}$.

(14) Let R be a ring and I be an ideal. Show that $F^{e*}(R/I) \cong R/I^{[p^e]}$.

(15) [†] Let W be a multiplicatively closed subset of R. Show that $F^e_*(W^{-1}R) \cong W^{-1}F^e_* R$.

(16) Let $K = \mathbb{F}_p(t_1, t_2, t_3, \dots)$, and $R = K[\![x]\!]$. Show that $F_* R$ is not a free module. Compare with Corollary 1.7.

(17) Let R be a ring and I be an ideal. Is $F^{e*}(I) \cong I^{[p^e]}$ in general?

(18) Let R be a ring containing $\mathbb{Q}$, let n be a positive integer, and I an ideal of R. Show that the ideal $(a^n \mid a \in I)$ is equal to I^n. Compare to problem (2).

(19) [†] Let R be a Noetherian ring of positive characteristic. Show that F_R is surjective if and only if R is a finite product of perfect fields.

(20) [†] Let R be an F-finite Noetherian ring. Show that the singular locus of R is a closed subset of $\mathrm{Spec}(R)$.

(21) [†] Let R be a regular Noetherian ring and M be a finitely generated module.
  (a) Show that $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(F^e_* M)$ for all e.
  (b) Show that $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(F^{e*} M)$ for all e.
  (c) Do the statements (21a) and (21b) hold if R is not assumed to be regular?

[§] To be used in Problem set #2.

[†] Requires some background from Commutative Algebra.

2. Tight closure

We now discuss a notion based on the Frobenius map that has many powerful applications.

Definition 2.1. Let R be a ring of characteristic p and $I \subseteq R$ be an ideal. The **Frobenius closure** of I is the ideal

$$I^F := \{a \in R \mid a^{p^e} \in I^{[p^e]} \text{ for some } e > 0\}.$$

Definition 2.2 (Hochster-Huneke). Let R be a domain of characteristic p and $I \subseteq R$ be an ideal. The **tight closure** of I is the ideal

$$I^* := \{a \in R \mid \exists c \neq 0 \ : \ ca^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\}.$$

When R is not necessarily a domain, we instead insist that c is not in any minimal prime ideal of R.

It follows from the definitions that $I \subseteq I^F \subseteq I^*$. These are notions that say that an element is in asymptotically in I, in various senses. The main fact about tight closure we will observe today is the following:

Theorem 2.3. *Let S be a polynomial ring over a perfect field K (or more generally, a regular ring of characteristic p). Then for any ideal $I \subseteq S$, we have $I^* = I$.*

The statement may look a bit odd, but the point of the theorem is that it can be much easier to check that an element is in I^* rather than I. We need a lemma to prepare for the proof.

Lemma 2.4. *Let $\varphi : A \longrightarrow B$ be a homomorphism of rings such that B is a free (or more generally, flat) A-module by restriction of scalars, and let I be an ideal of A, and $f \in A$. Then $(IB :_B f) = (I :_A f)B$.*

Proof. The containment $\supseteq$ follows from the definitions without assuming anything about B. For the other containment, let $g \in (IB :_B f)$, so there exist $a_i \in I$ and $b_i \in B$ such that

$$gf = \sum_i a_i b_i.$$

Let $\{\beta_j\}$ be a basis for B as an A-module, so we can write

$$g = \sum_j g_j \beta_j \quad b_i = \sum_j b_{ij} \beta_j$$

for some $g_j, b_{ij} \in A$. Then substituting in we get

$$\left(\sum_j g_j \beta_j\right) f = \sum_i a_i \left(\sum_j b_{ij} \beta_j\right)$$

$$\sum_j f g_j \beta_j = \sum_j \left(\sum_i a_i b_{ij}\right) \beta_j$$

Now, using the A-linear independence of β_j, we get equations of the form

$$f g_j = \sum_i a_i b_{ij},$$

so $g_j \in (I :_A f)B$; then since g is a B-linear combination of g_j, we have $g \in (I :_A f)B$. □

Proof of Theorem 2.3. We always have $I \subseteq I^*$ so there is only one containment left to show. Let $a \in I^*$, so there exists $c \neq 0$ with $ca^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. In particular, for $e \gg 0$ have

$$c \in (I^{[p^e]} :_S a^{p^e})$$

Let us consider the analogue of this same containment in $F_*^e R$:

$$F_*^e c \in (F_*^e(I^{[p^e]}) :_{F_*^e S} F_*^e(a^{p^e})) = (I F_*^e S :_{F_*^e S} a),$$

where we have applied the exercise. By the Lemma and Kunz' Theorem, we have

$$F_*^e c \in (I :_S a) F_*^e S = F_*^e\Big((I :_S a)^{[p^e]}\Big),$$

again using the exercise. That is,

$$c \in (I :_S a)^{[p^e]}.$$

If $a \notin I$, then $(I :_S a) \subsetneq S$ and

$$c \in \bigcap_{e \gg 0} (I :_S a)^{[p^e]} \subseteq \bigcap_{e \gg 0} (I :_S a)^{p^e} = 0,$$

a contradiction. Thus, we must have $a \in I$. ☐

Let us illustrate a typical application of tight closure. By way of motivation, let K be a field, and $R = K[x]$ be a polynomial ring in one variable. Given any two elements $f, g \in R$, we claim that $fg \in (f^2, g^2)$. To see it, let d be the GCD of f and g, and write $f = df'$ and $g = dg'$. Then f' and g' are coprime and R is a PID so we can find r, s with $rf' + sg' = 1$. Then

$$fg = d^2 f'g' = d^2 f'g'(rf' + sg') = rg'(d^2 f'^2) + sf'(d^2 g'^2) = rg'f^2 + sf'g^2 \in (f^2, g^2).$$

Now take a polynomial ring in two variables $K[x,y]$. The previous argument certainly fails since R is not a PID, and even more convincingly since

$$xy \notin (x^2, y^2).$$

The next best thing to hope for that for any $f, g, h \in K[x,y]$ we have $fgh \in (f^2, g^2, h^2)$. This is also false; we learned the following example from Anurag K. Singh:

$$(xy)(x^2 - y^2)(x^2 + y^2) \notin \Big((xy)^2, (x^2 - y^2)^2, (x^2 + y^2)^2\Big),$$

at least if K has characteristic other than two.

However, the next best thing is true: for any $f, g, h \in K[x,y]$ we have $f^2 g^2 h^2 \in (f^3, g^3, h^3)$.

Theorem 2.5. *Let K be a field, and $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over K. Then for any $f_1, \ldots, f_{n+1} \in S$, the containment*

$$f_1^n \cdots f_{n+1}^n \in (f_1^{n+1}, \ldots, f_{n+1}^{n+1})$$

holds.

We will prove this theorem in the case that K algebraically closed of characteristic p, and $n = 2$ just to keep notation simpler. One can in fact deduce the theorem for all fields from this case. The Theorem holds more generally when S is a regular local ring of dimension n, though it requires different techniques in mixed characteristic.

Lemma 2.6. *Let R be a local ring of dimension n with an infinite residue field, $f \in R$, and I be an ideal of R. If $f^s \in I^s$ for some s, then there exists c not in any minimal prime of R such that $cf^t \in (\ell_1, \ldots, \ell_n)^t$ for all $t \gg 0$, where $\ell_1, \ldots, \ell_n$ are n general linear combinations of the generators of I.*

The Lemma follows from some standard facts in integral closure theory; we outline a self-contained proof in the next exercise set.

Proof of Theorem 2.5 for K of positive characteristic. Standard reductions allow us to replace the polynomial ring with a regular local ring R of dimension n with an infinite residue field. Let's just do the case $n = 2$ for simplicity.

Let $f, g, h \in R$ a regular local ring of dimension two with infinite residue field. We need to show that $(fgh)^2 \in (f^3, g^3, h^3)$. Observe that $(fgh)^3 \in (f^3, g^3, h^3)^3$. We can apply the Lemma to get some $c \neq 0$ such that $c(fgh)^t \in (\ell_1, \ell_2)^t$ for $t \gg 0$. Now take $e \gg 0$ and set $t = 2p^e$:

$$\begin{aligned} c(fgh)^{2p^e} \in (\ell_1, \ell_2)^{2p^e} &\subseteq (\ell_1^{2p^e}, \ell_1^{2p^e-1}\ell_2, \dots, \ell_1^{p^e}\ell_2^{p^e}, \dots, \ell_1\ell_2^{2p^e-1}, \ell_2^{2p^e}) \\ &\subseteq (\ell_1^{p^e}, \ell_2^{p^e}) = (\ell_1, \ell_2)^{[p^e]} \subseteq (f^3, g^3, h^3)^{[p^e]}. \end{aligned}$$

We can rewrite this as

$$c\left((fgh)^2\right)^{p^e} \in (f^3, g^3, h^3)^{[p^e]}$$

for $e \gg 0$. This means that $(fgh)^2 \in (f^3, g^3, h^3)^*$. By Theorem 2.3, we deduce that $(fgh)^2 \in (f^3, g^3, h^3)$. The proof for $n > 2$ is similar. ☐

The last thing we want to illustrate is that statements over fields of characteristic zero can be deduced from statements in characteristic p. We will use the following facts from Commutative Algebra:

Lemma 2.7. *Let A be a finitely generated ring over $\mathbb{Z}$; for example a finitely generated subring of a field K. Then*

1. *For any maximal ideal $\mathfrak{m}$ of A, the quotient $A/\mathfrak{m}$ is a finite field.*
2. *For a polynomial ring $S = A[x_1, \dots, x_n]$, and element $f \in S$ and ideal $I \subseteq S$, if $f \in I + \mathfrak{m}S$ for every maximal ideal $\mathfrak{m}$ of A, then $f \in I$.*

This is all we need to deduce the Theorem in characteristic zero!

Proof of Theorem 2.5 for K of characteristic zero. We stick with f, g, h for simplicity. Suppose that we have $f, g, h \in K[x, y]$. Let A be the subring of K generated by the coefficients of f, g, h in K; this is a finite set, so A is a finitely generated ring, and $f, g, h \in A[x, y]$. Now let $\mathfrak{m}$ be a maximal ideal of A. Writing $\overline{*}$ for images modulo $\mathfrak{m}$, we have $\overline{f}, \overline{g}, \overline{h} \in A[x, y]/\mathfrak{m}A[x, y] \cong (A/\mathfrak{m})[x, y]$. Since $A/\mathfrak{m}$ is a field of characteristic zero, we have

$$(\overline{f}\overline{g}\overline{h})^2 \in (\overline{f}^3, \overline{g}^3, \overline{h}^3) \quad \text{in } (A/\mathfrak{m})[x, y].$$

This means that

$$(fgh)^2 \in (f^3, g^3, h^3) + \mathfrak{m}A[x, y] \quad \text{in } A[x, y].$$

Since this is true for all maximal ideals $\mathfrak{m}$, we deduce that

$$(fgh)^2 \in (f^3, g^3, h^3) \quad \text{in } A[x, y].$$

But since $A \subseteq K$, we obtain

$$(fgh)^2 \in (f^3, g^3, h^3) \quad \text{in } K[x, y].$$

☐

3. F-singularities

So far we have largely focused on advantageous properties of the Frobenius map when R is a polynomial ring, or more generally, a regular ring, in light of Kunz' theorem. Let us focus on the case of the polynomial ring over a perfect field or the case of an F-finite regular local ring. In either of these cases, F^e_*R is free over R. We have applied this in the setting of tight closure to say that there are "no new relations" in F^e_*R, which then led to triviality of tight closure. We will now consider the following perspective on freeness of F^e_*R: this means that F^e_*R has many surjective maps back to R, namely the coordinate maps for a free basis. We will consider weakenings of the conclusion of Kunz' theorem by asking for fewer surjective maps back to R.

Definition 3.1. Let R be a ring of characteristic p. We say that R is **F-split** if there is an R-module homomorphism $\varphi : F_*R \longrightarrow R$ such that $\varphi(F_*1) = 1$.

Example 3.2. Let K be a perfect field and $S = K[x_1, \dots, x_n]$. Recall that F_*R is a free R-module with basis $B = \{F_*(x_1^{a_1} \cdots x_n^{a_n}) \mid 0 \leq a_i < p\}$. Among this basis is F_*1. There is an S-linear map $\varphi : F_*S \longrightarrow S$ that sends any element $F_*s \in F_*S$ to the coefficient of F_*1 in the unique expresion of F_*s as an S-linear combination of the elements of B. In particular, $\varphi(F_*1) = \varphi(1 \cdot F_*1 + 0 \cdot \text{other elements of } B) = 1$. Thus S is F-split.

Example 3.3. Let K be a perfect field and $R = K[x,y]/(xy)$. We saw in the exercises that

$$F_*R \cong R \cdot F_*1 \oplus \bigoplus_{0<i<p} R/(y) \cdot F_*(x^i) \oplus \bigoplus_{0<j<p} R/(x) \cdot F_*(y^j).$$

An argument similar to the previous example shows that R is F-split.

Lemma 3.4. *An F-split ring is reduced.*

Proof. We will show that the Frobenius map $F : R \longrightarrow F_*R$ is injective. Let $\varphi : F_*R \longrightarrow R$ be an R-module homomorphism with $\varphi(F_*1) = 1$. Then for any $r \in R$,

$$\varphi F(r) = \varphi(F_*r^p) = \varphi(rF_*1) = r\varphi(F_*1) = r.$$

Thus, if $F(r) = 0$, then $r = \varphi F(r) = 0$ as well. This shows that F is injective, so R is reduced. □

There are a few useful equivalences for the F-split condition.

Lemma 3.5. *Let R be a ring of characteristic p. The following are equivalent:*

1. *R is F-split: there is an R-module homomorphism $\varphi : F_*R \longrightarrow R$ such that $\varphi(F_*1) = 1$.*
2. *For all $e > 0$, there is an R-module homomorphism $\varphi : F^e_*R \longrightarrow R$ such that $\varphi(F^e_*1) = 1$.*
3. *For some $e > 0$, there is an R-module homomorphism $\varphi : F^e_*R \longrightarrow R$ such that $\varphi(F^e_*1) = 1$.*
4. *For some $e > 0$, there is some $c \neq 0$ and an R-module homomorphism $\varphi : F^e_*R \longrightarrow R$ such that $\varphi(F^e_*c) = 1$.*

The implications (2)⇒(1)⇒(3)⇒(4) are clear. The rest are outlined in the exercises.

Lemma 3.6. *Let R be an F-split ring and I an ideal. Then $I^F = I$.*

Proof. Recall that $I^F = \{a \in R \mid a^{p^e} \in I^{[p^e]} \text{ for some } e\}$. Take some $a \in I^F$, so $a^{p^e} \in I^{[p^e]}$ for some e. We can rewrite this as

$$aF^e_*1 = F^e_*a^{p^e} \in F^e_*I^{[p^e]} = IF^e_*R,$$

so $aF^e_*1 = \sum_i a_i F^e_* r_i$ with $a_i \in I$. By the equivalences above, since R is F-split, we have a map φ such that $\varphi(F^e_*1) = 1$. We get

$$a = \varphi(aF^e_*1) = \varphi\left(\sum_i a_i F^e_* r_i\right) = \sum_i a_i \varphi(F^e_* r_i) \in I. \qquad \square$$

There is an extremely useful criterion for checking when a ring is F-split.

Theorem 3.7 (Fedder's criterion). *Let $(S, \mathfrak{m})$ be an F-finite regular local ring of characteristic p, and I an ideal of S. Then the ring S/I is F-split if and only if*

$$I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}.$$

The colon ideal $I^{[p]} : I$ is easy to compute in the special case when $I = (f)$ is a principal ideal; in this case $I^{[p]} : I = (f^{p-1})$. More generally, the colon ideal $I^{[p]} : I$ is easy to compute in the case that I generated by a regular sequence $f_1, \dots, f_t$. Recall that $f_1, \dots, f_t$ is a regular sequence if f_i is a nonzerodivizor modulo $f_1, \dots, f_{i-1}$ for each i. In this case $I^{[p]} : I = (f_1 \cdots f_t)^{p-1} + I^{[p]}$.

We will outline the proof of Fedder's criterion in the exercises.

Example 3.8. Let K be a field, and consider a 3×3 matrix

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

M is **nilpotent** if $M^n = 0$ for some n. For any given n, we can write out the nine entries M^n as polynomial expressions of the entries x_{ij} (of degree n) and we get nine equations to determine if $M^n = 0$. Much better, M is nilpotent if and only if the characteristic polynomial of M is of the form $T^3 = 0$, so the coefficients of the characteristic polynomial vanish. These are

$$f = x_{11} + x_{22} + x_{33}\ ,\ g = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} + \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix}\ ,\ h = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}.$$

One can see (e.g., from the next observation) that f, g, h form a regular sequence. Order the variables $x_{11} > x_{12} > x_{13} > x_{21} > \cdots > x_{33}$ and take the reverse lexicographic order on the polynomial ring. Then

$$LT((fgh)^{p-1}) = LT(f)^{p-1} LT(g)^{p-1} LT(h)^{p-1} = (x_{11}x_{12}x_{21}x_{13}x_{22}x_{31})^{p-1} \notin \mathfrak{m}^{[p]},$$

so the quotient ring is F-split.

In particular, the ideal generated by f, g, h is a radical ideal. While one can see this directly from initial ideal methods in this example, the combination of such methods with Fedder's criterion is a useful technique for showing an ideal is radical.

There is a stronger condition that is closely related.

Definition 3.9. Let R be a ring of characteristic p. We say that R is **strongly F-regular** if for any c not in any minimal prime of R, there is some e and an R-module homomorphism $\varphi : F^e_* R \longrightarrow R$ such that $\varphi(F^e_* c) = 1$. When R is a domain, this simplifies to: for any $c \neq 0$, there is some e and an R-module homomorphism $\varphi : F^e_* R \longrightarrow R$ such that $\varphi(F^e_* c) = 1$.

It follows from the definition that any strongly F-regular ring is F-split: one can enforce the definition with $c = 1$, and use the equivalences established above. If R is strongly F-regular and c not in any minimal prime of R, given an e that "works", any larger e also "works."

Lemma 3.10. *Let R be a strongly F-regular ring and I an ideal. Then $I^* = I$.*

Proof. Recall that

$$I^* = \{a \in R \mid \text{there exists } c \text{ not in any minimal prime } : \ ca^{p^e} \in I^{[p^e]} \text{ for } e \gg 0\}.$$

Take some $a \in I^*$, so $ca^{p^e} \in I^{[p^e]}$ for some c not in any minimal prime and $e \gg 0$. We can rewrite this as

$$aF^e_* c = F^e_*(ca^{p^e}) \in F^e_* I^{[p^e]} = IF^e_* R.$$

From the definition of strongly F-regular with c and the note above, for all $e \gg 0$ there is some $\varphi : F^e_* R \longrightarrow R$ such that $\varphi(F^e_* c) = 1$. Applying φ, we get

$$a = a\varphi(F^e_* c) = \varphi(aF^e_* c) = \varphi\left(\sum_i a_i F^e_* r_i\right) = \sum_i a_i \varphi(F^e_* r_i) \in I. \qquad \square$$

It is a longstanding open question whether a ring with the property that every ideal is tightly closed is necessarily strongly F-regular.

Proposition 3.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite regular local ring. Then R is strongly F-regular.*

Proof. The main point is the Corollary to Kunz' theorem: $F^e_* R$ is a free R-module for each e in this setting. Let $c \neq 0$. We also need a couple of standard facts from Commutative Algebra. First, the Krull Intersection Theorem says that $\bigcap_{n>0} \mathfrak{m}^n = 0$ in any local ring. Thus $\bigcap_{e>0} \mathfrak{m}^{[p^e]} \subseteq \bigcap_{e>0} \mathfrak{m}^{p^e} = 0$, so there is some e such that $c \notin \mathfrak{m}^{[p^e]}$. Second, a consequence of Nakayama's Lemma says that for M a finitely generated free module over a local ring $(R, \mathfrak{m})$, any element not in $\mathfrak{m}M$ is part of a free basis of M. Applying this to $F^e_* R$, we have $\mathfrak{m}F^e_* R = F^e_* \mathfrak{m}^{[p^e]}$. Thus, with e as above, $F^e_* c$ is part of a free basis for $F^e_* R$. Completing $\beta_1 = F^e_* c$ to a full basis $\{\eta_i\}$ for $F^e_* R$, there is an R-linear map φ that sends $\sum_i r_i \beta_i$ to r_1. In particular, $\varphi(F^e_* c) = 1$. $\square$

There is an analogue of Fedder's criterion, called Glassbrenner's criterion, for strong F-regularity. However, we will focus on another important source of strongly F-regular rings.

Definition 3.12. Let $R \subseteq S$ be an inclusion of rings. We say that R is a **direct summand** of S if there is an R-module homomorphism $\psi : S \longrightarrow R$ such that $\psi(1) = 1$.

Proposition 3.13. *Let $R \subseteq S$ be an inclusion of rings of characteristic p, and suppose that R is a direct summand of S.*

1. *If S is a strongly F-regular domain, then R is strongly F-regular.*
2. *If S is F-split, then R is F-split.*

Proof. We will prove the first statement, as the second is very similar. Let S be strongly F-regular, and $\psi : S \longrightarrow R$ such that $\psi(1) = 1$. Suppose that $c \neq 0$ in R. There is some e and S-linear map $\varphi : F^e_* S \longrightarrow S$ such that $\varphi(F^e_* c) = 1$. Since $R \subseteq S$, φ is R-linear as well. The restriction of the composition $\psi \circ \varphi|_{F^e_* R} : F^e_* R \longrightarrow R$ is an R-linear map sending $F^e_* c$ to 1. This shows that R is strongly F-regular. $\square$

Example 3.14. Let K be a perfect field. Let $R = K[x^2, xy, y^2] \subseteq S = K[x, y]$. We claim that R is a direct summand of S. Note that R is the K-vector space spanned by monomials whose total degree is even. Any element $s \in S$ has a unique expression of the form $s = s_{even} + s_{odd}$ where s_{even} is a linear combination of monomials of even degree, i.e., $s_{even} \in R$, and s_{odd} is a linear combination of monomials of odd degree. Thus, there is a well-defined map $\psi : S \longrightarrow R$ given by $\psi(s) = s_{even}$. This map is R-linear: if $r \in R$, then $rs = rs_{even} + rs_{odd}$, where rs_{even} is a linear combination of monomials of even degree and rs_{odd} is a linear combination of monomials of odd degree. This means that $\psi(rs) = rs_{even} = r\psi(s)$, which says that ψ is R-linear.

We now loosely outline an application of strong F-regularity. A **magic square** of size t with row sum n is a $t \times t$ array of nonnegative integers such that each row and each column sums to n. For example,

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

is a particularly gaudy magic square of size 4 and row sum 33. There is only one magic square of size 1 and row sum n, namely

$$\boxed{n}$$

and there are $n+1$ magic squares of size 2 and row sum n, namely

i	$n-i$
$n-i$	i

$$0 \leq i \leq n.$$

Theorem 3.15 (Stanley). *Denote by $M_t(n)$ the number of $t \times t$ magic squares with row sum n. For any $t > 0$, the function $M_t(n)$ is a polynomial for $n \geq 0$.*

Outline. Step 1: Let K be a field of positive characteristic and $S = K[x_{11}, \ldots, x_{33}]$. We associate to each magic square a monomial in S:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \rightsquigarrow \quad x_{11}^{a_{11}} x_{12}^{a_{12}} \cdots x_{33}^{a_{33}}$$

and we let R be the K-vector space spanned by these monomials. R is a subring of S. The number $M_t(n)$ is equal to the number of monomials in R of degree nt, or equivalently, the vector space dimension of R_{nt}. This part is elementary.

Step 2: The ring R is generated over K by magic squares of row sum 1; i.e., R is generated in a single degree. This boils down to the fact that any magic square is a sum of permutation matrices, which is a nontrivial fact from combinatorics/convex geometry called the Birkhoff-Von Neumann Theorem. If we divide all of the degrees in R through by t, then R is generated in degree one, and $M_t(n)$ is now just the Hilbert function of R. It follows from general facts that $M_t(n)$ *eventually* agrees with a polynomial, but we want to show that it agrees with a polynomial for all nonnegative values of n.

Step 3: The ring R is a direct summand of S. This is not too hard to show. It then follows that R is a strongly F-regular graded ring.

Step 4: The fact that R is strongly F-regular forces certain graded pieces of local cohomology to vanish, which then forces R to be Cohen-Macaulay and the regularity of R to be less than the dimension of R. These conditions then make the Hilbert function of R a polynomial. ☐

Exercise set #2

(1) Explain as succinctly as possible why the ring $K[x,y]/(x^2)$ is not F-split nor strongly F-regular.

(2) Let K be a perfect field of characteristic p and $S = K[x,y]_{(x,y)}$. Recall that this is an F-finite regular local ring. Apply Fedder's criterion to the rings $S/(x^2)$ and $S/(xy)$. Compare this to our other examples.

(3) Let K be a perfect field of characteristic p and $S = K[x,y,z]_{(x,y,z)}$. Apply Fedder's criterion to:
- $S/(x^2+y^2+z^2)$. It may be helpful to consider the cases with $p=2$ and $p \neq 2$ separately.
- $S/(x^4+y^4+z^4)$.
- $S/(x^3+y^3+z^3)$. It may be helpful to consider the cases with $p=3$, $p \equiv 1 \mod 3$, and $p \equiv 2 \mod 3$ separately.

(4) Let K be a field of characteristic $\neq 2$ and $S = K[x,y]$. Verify that $fgh \notin (f^2,g^2,h^2)$ for $f = xy$, $g = x^2-y^2$, $h = x^2+y^2$.

(5) Complete the proof of Lemma 3.5.

Hint: For (3)⇒(1)⇒(2), think of $R \xrightarrow{F^{e+e'}} F_*^{e+e}R$ as the composition $R \xrightarrow{F^e} F_*^eR \xrightarrow{F_*^{e'}F} F_*^{e+e'}R$. You may find it useful to show that if there is some e that "works" any smaller e "works", and if e "works", then $2e$ "works".

(6) Let K be a field, and $R \subseteq S = K[x_{11},\dots,x_{33}]$ be the K-vector space spanned by monomials $x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{33}^{a_{33}}$ such that $\{a_{ij}\}$ is a magic square. Explain why R is a ring, and show R is a direct summand of S via the K-vector space map $\psi : S \longrightarrow R$ given by

$$\psi(x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{33}^{a_{33}}) = \begin{cases} x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{33}^{a_{33}} & \text{if } \{a_{ij}\} \text{ is a magic square} \\ 0 & \text{otherwise.} \end{cases}$$

(7) Let K be a perfect field of characteristic p and $R = K[x,y,z]/(x^3+y^3+z^3)$.
  (a) If $p \equiv 2 \mod 3$, show that $(z^2)^p \in (x,y)^{[p]}$. Deduce that $z^2 \in (x,y)^F$ and $z^2 \in (x,y)^*$. Compare this with (3) above.
  (b) If $p \equiv 1 \mod 3$, show that $z^2 \in (x,y)^*$. Deduce that R is not strongly F-regular.

(8) † Lemma 2.6 follows from standard properties of integral closure, but we outline a self-contained argument in the case of polynomials $f_1,\dots,f_{n+1}$ homogeneous of the same degree in a polynomial ring $S = K[x_1,\dots,x_n]$ over an infinite field K.
  (a) Let T be an indeterminate. Explain why $K[f_1,\dots,f_{n+1}] \cong K[f_1T,\dots,f_{n+1}T] \subseteq R[T]$.
  (b) Let $f_1,\dots,f_{n+1} \in S$ be homogeneous polynomials of the same degree. Explain why the inclusion $K[\ell_1T,\dots,\ell_nT] \subseteq K[f_1T,\dots,f_{n+1}T]$ is module-finite for generic linear combinations $\ell_1,\dots,\ell_n$ of $f_1,\dots,f_{n+1}$.
  (c) Show that the inclusion $R[\ell_1T,\dots,\ell_nT] \subseteq R[f_1T,\dots,f_{n+1}T]$ is module-finite for generic $\ell_1,\dots,\ell_n$.
  (d) Take an equation $(f_iT)^k + \cdots = 0$ of integral dependence for f_iT over $R[\ell_1T,\dots,\ell_nT]$ and collect the terms of the form T^k. Use this to show that $f_i^{k+t} \in (\ell_1,\dots,\ell_n)^t$.

(9) † Let R be a strongly F-regular Noetherian local or graded ring. Show that R is a domain. Hint: If R has distinct minimal primes, start by finding nonzero f,g such that $fg=0$ and $f+g$ is not in any minimal prime.

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†Requires some background from Commutative Algebra.

(10) In this problem, we prove Fedder's criterion in the case of $R = S/I$ for $S = K[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ with K perfect. We will use the conclusion of (13) from Problem Set 1 in this setting.
  (a) Explain why every R-linear map $\varphi : F_*R \longrightarrow R$ is induced from a map $\psi : F_*S \longrightarrow S$ in the sense that $\varphi(\overline{s}) = \overline{\psi(s)}$, thinking of $F_*R = F_*S/F_*I$.
  (b) Let Φ be as in problem (13) from Problem Set 1 and $s \in S$. Show that $(F_*s \cdot \Phi)(S) \subseteq \mathfrak{m}$ if and only if $s \in \mathfrak{m}^{[p]}$.
  (c) Show that $(F_*s \cdot \Phi)(I) \subseteq I$ if and only if $s \in (I^{[p]} : I)$. Deduce Fedder's criterion.

(11) In the context of the previous problem, show that

$$\operatorname{Hom}_R(F_*R, R) \cong \frac{F_*(I^{[p]} : I) \cdot \operatorname{Hom}_S(F_*S, S)}{F_*I^{[p]} \cdot \operatorname{Hom}_S(F_*S, S)}.$$

(12) † Compute the degree of the polynomial $M_t(n)$ for every t.

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5. Homological Methods (C. Miller)

The graded minimal free resolution of a graded module was first introduced by Hilbert. Resolutions continue to be the source of many interesting research questions. This course introduced the basic concepts in the area, along with important invariants like Hilbert functions, Betti numbers and the Castelnuovo-Mumford regularity. This course was be taught by Claudia Miller (Syracuse)

5.1 Video Links

You can watch the original lectures using the following links:

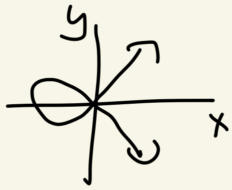
- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

5.2 Lecture Notes

We have included copies of Claudia's lecture notes and her tutorials.

L1: Graded free resolutions

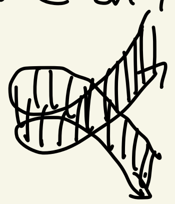
variety (vanishing set of polynomials) ⟷ coordinate ring (ring of poly fns on X)



$X = V(\underbrace{y^2 - x^2(x+1)}_{=0})$ ⟷ $R = \dfrac{k[x,y]}{(y^2-x^2(x+1))}$ ← 0 on X

sheaf $\mathcal{F}$ (structure on X) ⟷ R-module M

if locally free,



bundle ⟷ M projective (locally free)

To study: attach invariants

One of most important/basic are the free resolutions

[also: cohomology theories built from them]

Def: R commut. ring, M R-module

$$\cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

is a free resn of M if

- each F_i free R-mod ($F_i = R^{b_i}$)
- exact — $\operatorname{im}\partial_i = \ker\partial_{i-1}$
  - $\operatorname{im}\partial_1 = \ker\varepsilon$
  - ε surjective

Also write $F_\bullet \xrightarrow{\simeq} M$

∂ = "differential" = "bdry"

so, $M = \operatorname{im}\varepsilon = F_0/\ker\varepsilon = \left.F_0/\operatorname{im}\partial_1\right\}$ "coker ∂_1"

Construction:

① Let $m_1, \dots, m_{b_0}$ be a set of generators of M

Map a free of rank b_0 onto M:

$$\begin{tikzcd} F_0 = R^{b_0} \arrow[r, two heads, "\varepsilon"] & M \\ \text{basis } e_i \arrow[r, mapsto] & m_i \end{tikzcd}$$

$$e_i = (0,\dots,\underset{\substack{\uparrow \\ i\text{th}}}{1},\dots 0)$$

② Repeat for $\ker\varepsilon$:

$$\begin{tikzcd} F_1 = R^{b_1} \arrow[rr, "\partial_1 = i\pi"] \arrow[dr, two heads, "\pi"] & & F_0 \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0 \\ e'_i \arrow[dr, mapsto] & \underbrace{\ker\varepsilon}_{\text{say gen by } z_1,\dots,z_{b_1}} \arrow[ur, hook, "i"] & & & \\ & z_i & & & \end{tikzcd}$$

So, $\text{im}\,\partial_1 \underset{\substack{\uparrow \\ i \text{ inclusion}}}{=} \text{im}\,\pi = \ker\varepsilon$ (exact at F_0!)

Note: $\ker\varepsilon = \left\{ \sum r_i e_i \;\middle|\; \underbrace{\varepsilon(\sum r_i e_i) = 0}_{\sum r_i m_i = 0} \right\}$

$=$ the relations on the m_i's

③ Repeat for $\ker\pi = \ker\partial_1$ = relns among the relns!

$\vdots$ = relns among columns of matrix of ∂_1

$\partial_1(e'_i) = z_i$

$$\begin{tikzcd} \cdots \arrow[r] & R^{b_3} \arrow[rr] \arrow[dr, two heads] & & R^{b_2} \arrow[rr, "\partial_2"] \arrow[dr, two heads] & & R^{b_1} \arrow[rr, "\partial_1"] \arrow[dr, two heads] & & R^{b_0} \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0 \\ & & \ker\partial_2 \arrow[ur, hook] & & \ker\partial_1 \arrow[ur, hook] & & \ker\varepsilon \arrow[ur, hook] & & & \end{tikzcd}$$

If R local ($\mathfrak{m}$ = max) or $R = \bigoplus_{i=0}^{\infty} R_i$ std graded $\left(\begin{array}{l} R_0 = \text{field} \\ \mathfrak{m} = \bigoplus_{i>0} R_i \end{array} \right)$

Fact 1: a f.g. module F is free $\Leftrightarrow$ projective

Fact 2: M f.g (or f.g. graded)

TFAE i) choose min'l set of gens at each step

ii) $\operatorname{im} \partial_i \subseteq \mathfrak{m} F_{i-1}$

If so, $F_\bullet$ is unique up to iso.

Called "the min'l (gdd) free resolution (MFR)" of M.

Define ith Betti number $b_i = \operatorname{rank}_R F_i = \operatorname{rank}_k (\underbrace{F_i \otimes k}_{\partial = 0})$

$$= \dim_k \operatorname{Tor}^R(M, k)$$

Ex 1: $R = S = k[x,y]$

$I = (x^3, xy)$

Find MFR of S/I over S

$$\begin{tikzcd}
 & & & & M \arrow[d, equal] & \\
0 \arrow[r] \arrow[d] & S^1 \arrow[r, "\partial_2 = {\begin{bmatrix}-y\\x^2\end{bmatrix}}"] \arrow[d] & S^2 \arrow[r, "\partial_1 = {[x^3\ xy]}"] \arrow[d] & S^1 \arrow[r, two heads, "\pi"] & S/I \arrow[r] & 0 \\
0\ \cancel{\ker\partial_2} \arrow[ur] & \ker\partial_1 \arrow[ur] & \ker\pi \arrow[ur] & 1 \arrow[r, mapsto] & \bar{1} &
\end{tikzcd}$$

$e \mapsto$ $\ker\partial_1 = \left\langle \begin{bmatrix}-y\\x^2\end{bmatrix} \right\rangle$

e_1, e_2

$\ker\pi = I = \langle x^3, xy \rangle$

$e_1 \longmapsto x^3$

$e_2 \longmapsto xy$

$$\text{matrix } \partial_1 = [\underset{\substack{\uparrow\\ \partial(e_1)}}{x^3}\quad \underset{\substack{\uparrow\\ \partial(e_2)}}{xy}]$$

Compute:

- $\ker\partial_1 = \{ s_1e_1 + s_2e_2 \mid \underbrace{\partial_1(\sim\sim) = 0}_{s_1(x^3) + s_2(xy) = 0} \}$

<u>guess</u>: $s_1 = -y$, $s_2 = x^2$

$= \langle -ye_1 + x^2e_2 \rangle$

$= \left\langle \underset{\substack{\uparrow\\ \partial_2(e)}}{\begin{bmatrix}-y\\x^2\end{bmatrix}} \right\rangle \subseteq S^2$ in fact, generates ker!

- $\ker\partial_2 = \{ se \mid s\begin{bmatrix}-y\\x^2\end{bmatrix} = 0 \} = 0$

<u>Note</u>: Maps given by mult. by matrix $S^n \xrightarrow{A} S^m$

$\bar{v} \longmapsto A\bar{v}$

<u>Betti numbers</u>: $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 1$, $\beta_{\geq 3} = 0$ (for β_i, $i =$ "homological degree")

Graded free resns:

$S = k[x,y]$ is graded!

$= k \oplus k\{x,y\} \oplus k\{x^2, xy, y^2\} \oplus \cdots$

$= S_0 \oplus S_1 \oplus S_2 \oplus \cdots$

$\deg(S_j) = j$ "internal degree"

Want ∂_i to be degree 0 maps!

$$\begin{tikzcd}
0 \arrow[r] & Se \arrow[r, "\partial_2"] & \begin{matrix} Se_1 \\ \oplus \\ Se_2 \end{matrix} \arrow[r, "\partial_1"] & S^1 \arrow[r, "\pi"] & S/I \arrow[r] & 0
\end{tikzcd}$$

$1 \longmapsto 1$

deg3 $\nearrow$ $e_1 \longmapsto x^3$ deg 3

deg2 $\nearrow$ $e_2 \longmapsto xy$ deg 2

deg4 $\nearrow$ $e \longmapsto \underbrace{-ye_1}_{\substack{\deg 1+3 \\ =4}} + \underbrace{x^2e_2}_{\substack{2+2 \\ =4}}$

Equivalently:

$$0 \to S(-4) \xrightarrow{\begin{bmatrix} -y \\ x^2 \end{bmatrix}} \begin{matrix} S(-3) \\ \oplus \\ S(-2) \end{matrix} \xrightarrow{\begin{bmatrix} x^3 & xy \end{bmatrix}} S(0)$$

where $S(-n)$ means S with internal deg shifted $+n$

$$\left(S(-n)_m = S_{m-n} \right)$$

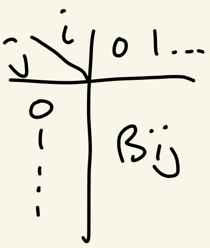
$$\left(\underset{\substack{\uparrow \\ \deg 0}}{0} \oplus \cdots \oplus 0 \oplus \underset{\substack{\uparrow \\ \deg n}}{S_0} \oplus S_1 \oplus \cdots \right)$$

Betti tables:

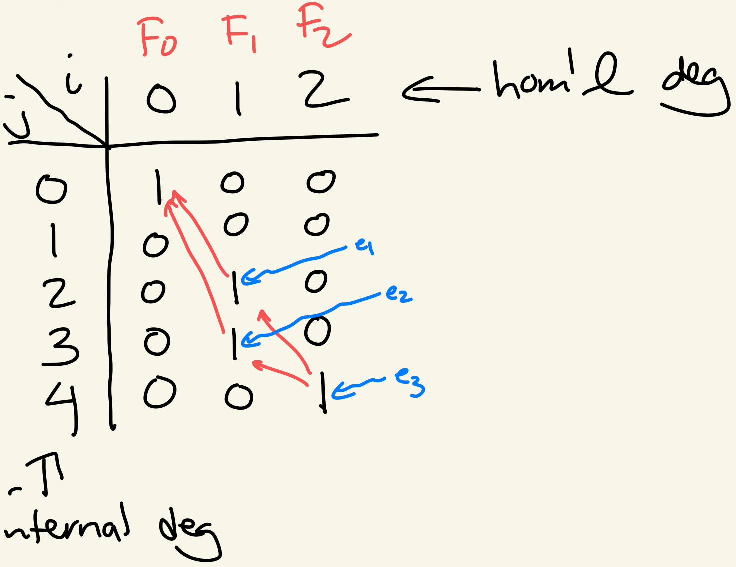
write $F_i = \bigoplus_{j\geq 0} S(-j)^{\beta_{ij}}$ $\left[\beta_{ij} = \dim_k \overset{\text{if min'l}}{\left(F_\cdot \otimes k\right)_j} = \dim_k \mathrm{Tor}^i(M,k)_j\right]$

the ranks β_{ij} called "graded Betti numbers"

1st attempt:



Ex:



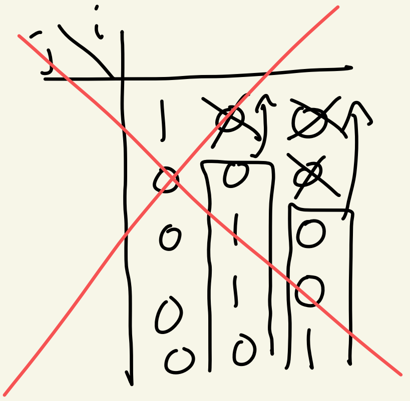
But: $F_\cdot$ min'l $\Rightarrow \partial(F_i) \subseteq \mathfrak{m} F_{i-1}$

$\Rightarrow$ matrix entries $\in \mathfrak{m}$

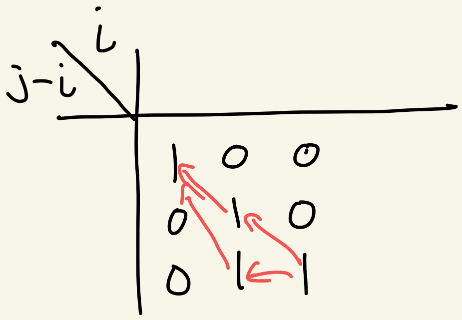
so, deg > 0

Ie, must shift at least 1 each step

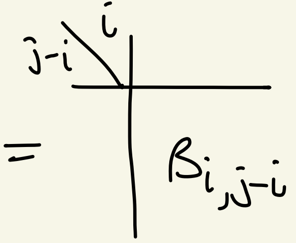
Ex:



$\rightsquigarrow$



"graded Betti table"



notice: linear parts of maps (es, $\partial(e) = \underset{\uparrow \text{ linear}}{-ye_1} + x^2 e_2$)

are horizontal rows

"linear strands"

L2: Taylor resolutions

Ex: $S = k[x,y]$ Find MFR of $S/(x^2, xy)$

$$0 \longrightarrow S(-3) \xrightarrow{\begin{bmatrix}-y\\ x\end{bmatrix}} S^{2}(-2) \xrightarrow{[x^2\ xy]} S^1 \to S/I \to 0$$

$$e_1 \longmapsto x^2$$
$$e_2 \longmapsto xy$$

obvious reln
($\partial(e_1 e_2)$ so, called "koszul reln") $\longrightarrow$ $-\cancel{x}ye_1 + x^{\cancel{2}}e_2 \longmapsto -xy(x^2) + x^2(xy)$
$= 0$

divide gcd $\;\underbrace{-ye_1 + xe_2}_{\text{generates kernel!}} \longmapsto 0$

Idea: min'l relns are $\dfrac{\text{koszul reln}}{\gcd} = -\dfrac{\text{lcm}}{x^2}e_1 + \dfrac{\text{lcm}}{xy}e_2$

In general:

$S = k[x_1, \dots, x_n]$

$I = (\underbrace{m_1, m_2, \dots, m_r}_{\text{monomials}})$

$F = S^r$ free module, basis $e_1, \dots, e_r$

The Taylor resolution is a modification of the koszul cx $K(m_1, \dots, m_r; S)$

w/ same free modules

$\nearrow$ not exact if $\exists \gcd(\text{pair}) \neq 1$

$$\begin{tikzcd}
T = (0 \arrow[r] & \Lambda^r F \arrow[r] & \cdots \arrow[r] & \Lambda^2 F \arrow[r] \arrow[d, equal] & \Lambda^1 F \arrow[r] \arrow[d, equal] & \Lambda^0 F \arrow[r] \arrow[d, equal] & 0 \\
 & & & \bigoplus_{i<j} Se_ie_j & F \arrow[d, equal] & S & \\
 & & & & \bigoplus Se_i & &
\end{tikzcd}$$

$\rightsquigarrow$

but with differential

- notation: for subset $I = \{i_1 < \cdots < i_p\} \subseteq \{1, \ldots, n\}$
  write $e_I = e_{i_1} \cdots e_{i_p}$
  $m_I = \mathrm{lcm}(m_{i_1}, \ldots, m_{i_p})$

- $$\partial(e_I) = \sum_{k=1}^{p} (-1)^{k+1} \frac{m_I}{m_{I \setminus \{i_p\}}} e_{I \setminus \{i_p\}}$$

$$= \sum (-1)^{k+1} \frac{\mathrm{lcm}(m_{i_1}, \ldots, m_{i_p})}{\mathrm{lcm}(m_{i_1}, \ldots, \widehat{m_{i_k}}, \ldots)} e_{i_1} \cdots \widehat{e_{i_k}} \cdots e_{i_p}$$

Thm [Taylor '66]

T is an S-free resn of S/I

[Gemeda '76 (also Fröberg '78): ∃ chain map $T \otimes T \to T$ making $\bigoplus T_i$ a gdd algebra } "dg algebra"]

$$e_I \cdot e_J = \begin{cases} \gcd(m_I, m_J)\, e_{I \cup J} & , \text{if } I \cap J = \emptyset \\ 0 & , \text{if } I \cap J \neq \emptyset \end{cases}$$

L3: CM regularity

Natural Q about Betti tables? bounds on right & bottom

Hilbert Syzygy Thm (1890)

M f.g module / $S = k[x_1, \dots, x_n]$

$\exists$ free resn $0 \to F_k \to F_{k-1} \to \dots \to F_0 \to M \to 0$

In fact • any MFR has same length, called projdim M or pd M

• any free resn $G_\cdot \xrightarrow{\sim} M$ has $\ker \partial_{n-1}$ free

So, can truncate $0 \to \ker \partial_{n-1} \to G_{n-1} \xrightarrow{\partial} \dots \to G_0 \xrightarrow{\varepsilon} M \to 0$

Def: ith syzygy $\operatorname{syz}_i M = \Omega_i(M) \overset{\text{def}}{=} \operatorname{im} \partial_i$ $(= \ker \partial_{i-1}$ or $\varepsilon)$

Thm [Auslander-Buchsbaum-Serre][ABS]

(R, m) local or gdd

$\underbrace{R \text{ regular}}_{}$ $\iff$ $\forall M$, $\operatorname{pd}_R M < \infty$

m gen'd by dim R elts $\iff$ $\operatorname{pd}_R k < \infty$

($\rightsquigarrow$ X nonsingular
dim tangent space
$=$ dim X)

CM regularity

$S = k[x_1,\dots,x_n] = \bigoplus_{i\geq 0} S_i$ std graded ($\deg x_i = 1$)

$M = \bigoplus_{\underbrace{i\geq 0}_{\text{or } i\geq -N}} M_i$ f.g gdd S-mod ($S_i \times M_j \xrightarrow{\mu} M_{i+j}$)

(eg, $M = S/I$ where $I = (\underbrace{f_1,\dots,f_r}_{\text{homogeneous}})$)

MFR: $0 \to \bigoplus_j S(-j)^{\beta_{nj}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0j}} \to M \to 0$

recall: shifts increase by at least 1 each step

Def: the Castelnuovo-Mumford (CM) regularity is the worst cumulative shift beyond expected min

$\operatorname{reg}_S M = \sup\{j-i \mid \beta_{ij}\neq 0\}$ = index of lowest row $\neq 0$ of Betti table, if it's finite

Ex:

$j-i$ \ i	0	1	2
0	1	0	0
1	0	1	0
2	0	1	1

$\leftarrow \operatorname{reg}_S S/I = 2$

Note: Can define $\operatorname{reg}_R M$ over any graded ring R, but:

Thm: [Avramov-Peeva, 01] R f.g gdd, $R_0 = k$ field

$\operatorname{reg}_R k < \infty \iff \forall M, \operatorname{reg} M < \infty$

$\iff R$ is a Koszul algebra with variables ($\deg > 1$) adjoined

} Av-Eisenbud conj'd

Thm [Eisenbud-Goto '84]

$$\operatorname{reg}_S M \; \left(\overset{\text{def}}{=} \max\{\overbrace{j-i}^{\ell} \mid \beta_{ij} \neq 0\}\right) \left(= \max\{\ell \mid \beta_{i,i+\ell} \neq 0\}\right)$$

$$= \min\{\ell \mid M_{\geq \ell} \text{ has a linear resn } 0 \to \cdots \to \oplus S(-\ell-1) \to \oplus S(\ell) \to M_{\geq \ell} \to 0\}$$

[note: as sheaves on $\mathbb{P}^n$, $\widetilde{M_{\geq \ell}} = \widetilde{M}$]

$$= \max\{\overbrace{j+i}^{\ell} \mid H^i_{\mathfrak{m}}(M)_j \neq 0\} = \max\{\ell \mid H^i_{\mathfrak{m}}(M)_{\ell-i} \neq 0\}$$

Local Cohomology [Grothendieck] $(R, \mathfrak{m})$

Def: $H^0_{\mathfrak{m}}(M) = \mathfrak{m}\text{-torsion of } M = \{x \in M \mid \mathfrak{m}^i x = 0, \text{ some } i\}$

Def: $H^i_{\mathfrak{m}}(M) = H^i(H^0_{\mathfrak{m}}(E^\bullet))$ where $M \to E^0 \to E^1 \to \cdots$ resn by injective mods

"right derived functor of $H^0_{\mathfrak{m}}(-)$"

$$= \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{m}^t, M)$$

$$\cong \varinjlim_t H^i(x_1^t, \ldots, x_n^t; M) \qquad \mathfrak{m} = \sqrt{(x_1, \ldots, x_n)}$$

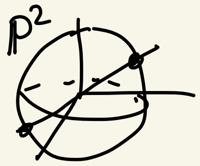
$$= H^i(\check{\text{C}}\text{ech complex } \check{C}(\underline{x}; M))$$

Explore in Advanced Pblms ...

Motivation / History

algebra

geometry



$S = k[x_0, \dots, x_n] \longleftrightarrow$ projective space

$\mathbb{P}^n = \dfrac{k^{n+1} \setminus \{\bar{0}\}}{\text{scaling}}$

$= \{\text{lines through } \bar{0}\}$

$=$ compactification of $k^n = \mathbb{A}^n$

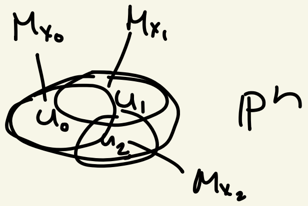
= affine space

homog ideal $I = (f_1, \dots, f_r) \rightsquigarrow V(I) = \{ \underline{x} \mid f_i(\underline{x}) = 0 \ \forall i \}$

$\rightsquigarrow \tilde{I}$ ideal sheaf

f.g. grd module $M = \oplus M_i \rightsquigarrow \tilde{M}$ coherent sheaf

$M_{\geq \ell} = M_\ell \oplus M_{\ell+1} \oplus \dots \rightsquigarrow$



"best M" $= \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\underbrace{\mathcal{F}(n)})$ $\leftsquigarrow$ $\mathcal{F} = \tilde{M}$

(saturated...)

global sections (module attached to $U = \mathbb{P}^n$)

where shifted mod: twisted sheaf:

$S(n) \longleftrightarrow \widetilde{S(n)} = \mathcal{O}_X(n)$

$M(n) \longleftrightarrow \widetilde{M(n)} = \mathcal{F}(n)$

$H^0(\mathbb{P}^n, \mathcal{F}) = \Gamma(\mathbb{P}^n, \mathcal{F})$

$H^i(\mathbb{P}^n, \mathcal{F}) =$ derived functor (use inj. resn of $\mathcal{F}$)

Serre Vanishing Thm [FAC, '55] ($\mathcal{F}$ coherent shf) (really on any proj. scheme w/ample line bdle $\mathcal{O}(1)$)

$$H^i(\mathbb{P}^n, \mathcal{F}(n)) = 0 \quad \underbrace{\text{for } n >> 0}_{\text{but when starts ??}}$$

Def [Mumford '66, based on Castelnuovo 1893]

$$\operatorname{reg}(\mathcal{F}) = \min\{r \mid H^i(\mathbb{P}^n, \mathcal{F}(r-i)) = 0, \ \forall i > 0\}$$

Facts/Uses: set $\begin{bmatrix} r = \operatorname{reg}(\mathcal{F}) \\ M = \Gamma_*(\mathcal{F}) \quad (\text{so, } \tilde{M} = \mathcal{F}) \end{bmatrix}$

1) $\underbrace{\dim_k H^0(\mathcal{F}(m))}_{\text{Hilbert fn}} = \text{poly for } m \geq r$

$= \dim M_m$ for $M = \Gamma_*(\mathcal{F})$

2) $\mathcal{F}(r)$ gen'd by global sections

3) under mild hypotheses (depth $M > 0$)

$r = \operatorname{reg}_S M$

4) by Eisenbud-Goto, linear resn (by line bdls)

$$0 \to \cdots\cdots \to \mathcal{O}_X(-r-1) \to \mathcal{O}_X(-r) \to \mathcal{F} \to 0$$

Thm: (classic, see [Ha]) for any M: $\mathcal{F} = \tilde{M}$

$$\bigoplus_n H^i(\mathbb{P}^n, \mathcal{F}(n)) = H^{i+1}_m(M) \quad \text{for } i \geq 1$$

nd

$$0 \to H^0_m(M) \hookrightarrow M \to \bigoplus_n \underbrace{H^0(\mathbb{P}^n, \mathcal{F}(n))}_{\Gamma(\mathcal{F})} \to H^1_m(M) \to 0$$

via Čech cohomology, a Meier-Vietoris seq, and

$$H^i(\mathbb{P}^n, \mathcal{F}) \simeq H^i(\underbrace{\mathbb{A}^n \setminus \{\bar{0}\}}_{U}, \mathcal{F})$$

5.3 Tutorial Problems

Here are the associated tutorial problems.

Problem Set 1 • Minimal Free Resolutions • Claudia Miller

Regular Problems:

1. Construct minimal graded free resolutions of each the following modules over the given ring. Indicate the shifts, and write the Betti table for each.

a) $S = k[x,y]$ and $M = S/(x^3, xy, y^3)$

b) $R = k[x,y]/(x^2y)$ and $M = R/(x)$

c) $R = k[x,y]/(x^3 + y^3)$ and $M = k = R/\mathfrak{m}$

d) $S = k[w,x,y,z]$ and $M = I = (wy - x^2, wz - xy, xz - y^2)]$ ← Try this one at home.

2. Which of the following Betti tables are impossible from general principles as *minimal* resolutions of a quotient of a polynomial ring?

	0	1	2	3	4
total:	1	6	6	2	1
0:	1	.	.	.	.
1:	.	4	5	2	.
2:	.	2	1	.	.
3:	.	.	.	.	1

	0	1	2	3	4
total:	1	8	6	0	1
0:	1	.	.	.	.
1:	.	6	5	.	.
2:	.	2	1	.	.
3:	.	.	.	.	1

	0	1	2	3	4
total:	1	6	6	2	1
0:	1	.	.	.	.
1:	.	4	.	2	.
2:	.	2	6	.	.
3:	.	.	.	.	1

3. a) Write the minimal free resolution of $k = S/\mathfrak{m}$ over $S = k[x_1, x_2]$.

b) Write out the Koszul complex $K(f, g; S)$ for arbitrary $f, g \in S$. (See definition below.)

c) Match your resolution from part (a) with a Koszul as in part (b).

d) Write out the Koszul complex $K(f_1, f_2, f_3; S)$ for arbitrary f_1, f_2, f_3 in a ring S.

Definition. Given elements $f_1, \ldots, f_r$ of any ring S, define the Koszul complex as follows:

- Let $F = S^r$ be the free R-module with basis $e_1, \ldots, e_r$
- Let $\bigwedge^p F$ be its pth exterior power.
  This is the free module of rank $\binom{r}{p}$ with basis given by formal products
  $$\{e_{i_1} \cdots e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq r\}$$
  of subsets of cardinality p of the basis of F.
- Then the Koszul complex $K(f_1, \ldots, f_r; S)$ on $f_1, \ldots, f_r$ is the S-free complex
  $$0 \to \bigwedge^r F \to \cdots \to \bigwedge^1 F \to \bigwedge^0 F$$
  with differential given on the basis by
  $$\partial(e_{i_1} \cdots e_{i_p}) = \textstyle\sum_{j=1}^{p} (-1)^{j+1} f_j \, e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_p}.$$

Definition. Recall $f_1, \ldots, f_r \in \mathfrak{m}$ is a regular sequence if $(f_1, \ldots, f_r \in \mathfrak{m}) \neq S$, f_1 is a nonzerodivisor on S, and for each $i > 1$ the element f_i is a nonzerodivisor on $S/(f_1, \ldots, f_{i-1})$.

Fact. Given homogeneous elements $f_1, \ldots, f_r$ of any graded ring S, the Koszul complex $K(f_1, \ldots, f_r; S)$ is acyclic (exact in positive homological positions) if and only if $f_1, \ldots, f_r$ is a regular sequence on S.

4. Try out some Macaulay2 code, as described at the end of this document.

Advanced Problems:

5. Prove Fact 1 and Fact 2 from Lecture 1.

Hint: Nakayama's Lemma – there are many versions, for both local and graded settings.

Here are some versions over a standard graded ring R.

Let I be a proper graded ideal in R and M be a finitely generated graded S-module.

- If $M = IM$, then $M = 0$.
- If elements $m_1, \dots, m_r \in M$ are such that their images generate M/IM, then they generate M. (And, if minimally for M/IM, then minimally for M.)
- If N is a submodule of M such that $M = N + IM$, then $M = N$.

These are especially useful for $I = \mathfrak{m}$.

6. Prove the Comparison Theorem and some consequences.

Let M and N be R-modules with projective resolutions $F \xrightarrow{\simeq} M$ and $G \xrightarrow{\simeq} N$.

(a) For any R-homomorphism $f \colon M \to N$, there exists a chain map $\widetilde{f} \colon F \to G$ lifting f, that is, with $\mathrm{H}_0(\widetilde{f}) = f$. (And it is unique up to homotopy, but you may skip this.)

(b) If $M = N$, then F and G are homotopy equivalent. (That is, there are chain maps between them whose compositions are homotopic to the identity maps.)

(c) Suppose R is local (or graded) and M is finitely generated (and graded).
If F and G are minimal resolutions of M, then F and G are isomorphic.
Hint: Nakayama's Lemma.
(Hence minimal resolutions are unique up to homotopy.)

Computing resolutions using the software system Macaulay2

This program uses Gröbner bases, which you will learn about from Fred's lectures this week!

- Go to the web site `macaulay2.com`.
- To run code, you may use their online interactive interface:
  Click on Macaulay2Web (on the left, just under Try It Out).
- Click on the start/play button at upper right.
- Type in the code – or type into a text file and cut-and-paste to the interface.
  item To restart the program hit the reset button at top (or type restart).

(Or you may download the program onto your computer/laptop from the website.)

Here is some Macaulay code for resolutions

```
S=QQ[x,y,z]
I=ideal(x^5+y^5)
R=S/I
A=matrix{{x,y}}
M=coker A
F=res(M, LengthLimit=>6)
betti F
```

You can search the documentation for further useful commands.

Problem Set 2 • Taylor resolution and CM Regularity • Claudia Miller

Regular Problems:

1. Construct the Taylor resolutions of each of the following.
Which are minimal?

a) $R = k[x, y, z]/(x^2, xyz^2, z^3)]$

b) $R = k[x, y, z, w]/(xyz, yzw)]$

c) $R = k[x, y, z, w]/(xy, yz, zw)]$ ← Try this one at home.

Instead of standard gradings (in $\mathbb{N}$), one can also use multi-grading (in $\mathbb{N}^m$ for some m).
Rewrite the shifts in the resolution you found **in part (b)** for the following multi-gradings.

- $\deg x = (1, 0, 0, 0),\ \deg y = (0, 1, 0, 0),\ \deg z = (0, 0, 1, 0),\ \deg w = (0, 0, 0, 1)$
- $\deg x = (1, 0),\ \deg y = (1, 0),\ \deg z = (0, 1),\ \deg w = (0, 1)$

2. Determine the regularity of each example in #1.

3. Often the CM regularity of a module is achieved at the last step of a resolution.
Give an example to show that it need not be.
Hint: You can do this problem even as a beginner. (Not a complicated construction.)

4. Here is an example for #3 that is of the form $M = S/I$. Find its Betti table using Macaulay2.
By the way, its resolution is also characteristic-dependent (try char 0 and 2).

```
I=ideal(x_1*x_2*x_3, x_1*x_2*x_4, x_1*x_3*x_5, x_1*x_4*x_6, x_1*x_5*x_6,
        x_2*x_3*x_6, x_2*x_4*x_5, x_2*x_5*x_6, x_3*x_4*x_5, x_3*x_4*x_6)
```

And here's a characteristic 0 example from Henry:

```
I=ideal(x*y+y*u,y^2-y*u,t*z,t*v,x*z*v,x*y*t)
```

5. Prove that for any ideal I in a ring S, one has $\operatorname{reg}_S I = \operatorname{reg}_S S/I + 1$.

6. Determine the regularity of the residue field $k = R/\mathfrak{m}$ for each ring below.

a) $R = k[x, y]/(xy)$

b) $R = k[x, y]/(x^3 + y^3)$ (you resolved this one last time)

7. Let M be a finitely generated S-module. Prove that

$$\beta_i(M) = \dim_k \operatorname{Tor}_i^S(M, k) \quad \text{and so} \quad \beta_{ij}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_j.$$

8. Let I be an ideal in $S = k[x_1, \ldots, x_n]$. Prove that its regularity is at most the regularity of its initial ideal. Hint: Gröbner deformation.

9. Try out some regularity experiments on Macaulay2 and make some conjectures.

over please...

Advanced Problems:

10. Prove the theorem by Eisenbud and Goto on regularity over $S = k[x_1, \dots, x_n]$.

(a) For the equivalence with the truncated module having a linear resolution:
Use that $\beta_{ij}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_j$ and compute the Tor module by using the Koszul complex $K(x_1, \dots, x_n; S)$ to resolve k.

(b) For the equivalence with local cohomology:
First, by truncating and shifting, we may assume that $\operatorname{reg}_S M = 0$ (so M has its generators in degree 0 and M has a linear resolution). Then:

(1) To get from Betti number vanishing to local cohomology, use induction on the Krull dimension of M together with the short exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \to M \to \overline{M} \to 0$$

to get a nonzero divisor on M and its associated short exact sequence.

(2) For the reverse direction, use Grothendieck duality:

$$\mathrm{H}^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}_S^{d-i}(M, S)^{\vee}$$

where $d = \dim S$, $(-)^{\vee} = \operatorname{Hom}_S(-, E)$, and $E = E(K)$ is the injective hull of k over S.

III

Week 2: Advanced Topics Lectures

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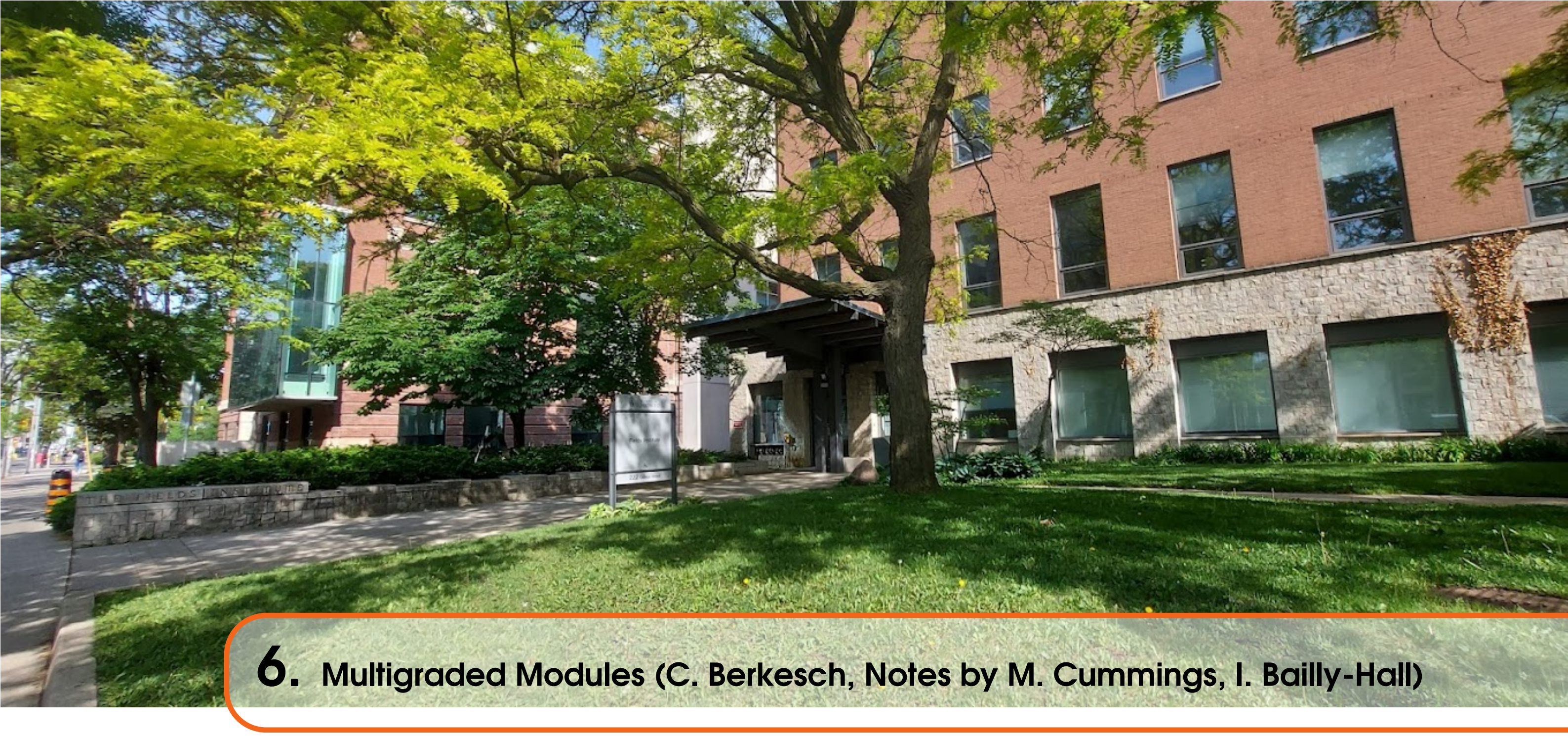


6. Multigraded Modules (C. Berkesch, Notes by M. Cummings, I. Bailly-Hall)

In this follow up course on resolutions, we discussed recent techniques and progress in the study of multigraded modules. In this context, we get “finer” invariants, like a multi-graded version of the Castelnuovo-Mumford regularity. This course was taught by Christine Berkesch (Minnesota).

6.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

6.2 Lecture Notes

We have included copies of Christine’s lecture notes and her tutorials. Lecture notes were provided by Mike Cummings and Isidora Bailly-Hall.

Multigraded modules (w/ Christine Berkesch) Reference Cox-Little-Schenck §5.2.

$\mathbb{C}$ = alg. closed field, $\pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n$. $S = \mathbb{C}[x_0, \dots, x_n]$, $m = \langle x_0, x_1, \dots, x_n \rangle$

$\boxed{\text{Everything is graded}}$

Def^n A (gld) <u>MFR</u> is a cx $F_\bullet : F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_\ell} F_\ell \leftarrow 0$ s.t. $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, $H_i F = 0$ $\forall i \geq 1$, & $\operatorname{im} \partial_i \subseteq m F_{i-1}$.

<u>Thm (Hilbert's Syzygy Theorem)</u>. $X \subseteq \mathbb{P}^n$, $I(X) \subseteq S$ ideal of X. The MFR of $S/I(X)$ has length $\leq n = \dim \mathbb{P}^n$.

Recommendation: Eisenbud's Geometry of Syzygies.

originally for showing existence of Hilb. poly.

<u>Notation</u> $S = \mathbb{C}[x_1, \dots, x_N]$.

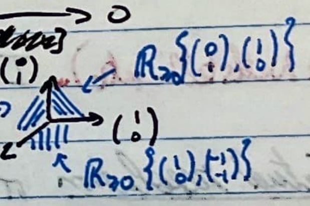
Def^n A <u>multigrading</u> on S is an assignment $\deg(x_i) := a_i \in \mathbb{Z}^d$

Eg $\mathbb{P}^1 \times \mathbb{P}^1$, $S = \mathbb{C}[x_0, x_1, y_0, y_1]$, $\deg(x_i) = (1,0)$, $\deg(y_i) = (0,1)$ $\in \mathbb{Z}^2$ $i = 0,1$.
homogeneous: $f = x_0^2 y_0^2 + x_0 x_1 y_1^2 + x_1^2(y_0^2 + y_0 y_1)$, degree $\binom{2}{2}$

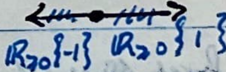
Std. grading on $\mathbb{P}^2$ - degree matrix $[1\ 1\ 1]$, now resolve this:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow[\begin{bmatrix}1&-1\\1&\cdot\\\cdot&1\end{bmatrix}]{} \mathbb{Z}^3 \xrightarrow[{[1\,1\,1]}]{} \mathbb{Z} \xrightarrow[\text{deg}]{} 0$$

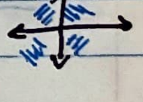
look at rows of S & take gen nonneg.



for $\mathbb{P}^1$: $0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix}-1\\1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{[1\ 1]} \mathbb{Z} \to 0$



$\mathbb{P}^1 \times \mathbb{P}^1$: $0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&\cdot\\-1&\cdot\\\cdot&1\\\cdot&-1\end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix}1&1&\cdot&\cdot\\\cdot&\cdot&1&1\end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0 \rightsquigarrow$



Also, $\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{C}^4 \setminus \mathbb{V}(\langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle)) / (\mathbb{C}^*)^2$
$(\mathbb{C}^*)^2$ action: $(\mathbb{C}^*)^2 \times \mathbb{C}^4 \to \mathbb{C}^4$ by $(s,t) \cdot (x_0, x_1, y_0, y_1) = (sx_0, sx_1, ty_0, ty_1)$

Def^n A <u>fan</u> Σ in $\mathbb{R}^n$ is a finite collection of cones σ ($\mathbb{R}_{\geq 0}\{$finitely-many vecs$\}$) s.t.
(a) every σ is contained in an open halfspace, (b) every subcone is in Σ
(c) cones in Σ closed under intersection.
The <u>support</u> of Σ is $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$. Write $\Sigma(r) = \{r\text{-dim'l cones}\}$

<u>Convention</u>. ("Simplicial") All cones gen'd by $\mathbb{R}$-LI vectors.

\# rays
these $|\cdot|$'s are cardinality of $\Sigma(1)$

Σ : a simplicial fan in $\mathbb{R}^d$, & $N := |\Sigma(1)|$

columns ⟷ variables

construct ses

construct this to make the seq. exact given Σ

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^d \arrow[r, "{\left[\begin{smallmatrix}\text{rays of } \Sigma \\ \text{as rows}\end{smallmatrix}\right]}"] & \mathbb{Z}^N \arrow[r, "{\left[\begin{smallmatrix}\text{\# grading} \\ \text{on } S\end{smallmatrix}\right]}"] & \mathbb{Z}^r \arrow[r] & 0 \end{tikzcd}$$

$S = \mathbb{C}[x_0, x_2, \dots, x_N]$

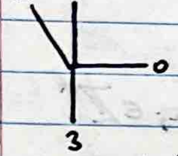
$S \supseteq B := \langle x^{\hat{\sigma}} := \prod_{i \notin \sigma} x_i \mid \sigma \in \Sigma \rangle$ "irrelevant ideal"

action given by the degree matrix $\mathbb{Z}^n \to \mathbb{Z}^r$

Def^n A simplicial toric variety $X = X_\Sigma$ is $\overset{\circ}{=}$ $(\mathbb{C}^N \setminus \mathbb{V}(B)) / (\mathbb{C}^*)^r$

Eg $X = \mathcal{H}_2$

not a product of proj. spaces



$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "{\left[\begin{smallmatrix}1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & -1\end{smallmatrix}\right]}"] & \mathbb{Z}^4 \arrow[r, "{\left[\begin{smallmatrix}1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1\end{smallmatrix}\right]}"] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd}$$

so $S = \mathbb{C}[x_0, \dots, x_3]$

secretly a $\mathbb{P}^1$ bundle

degrees: $\deg(x_0) = \binom{1}{0}$, $\deg(x_1) = \binom{-2}{1}$, $x_2 \mapsto \binom{1}{0}$, $x_3 \mapsto \binom{0}{1}$

action: $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^4$ by $(s,t) \cdot \vec{x} = (s x_0, s^{-2} t x_1, s x_2, t x_3)$

There is a 1-to-1 correspondence

$$\left\{ \begin{matrix} \text{closed subvarieties} \\ \text{of } X = X_\Sigma \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{homog. radical} \\ \text{ideals } I \subseteq B \subseteq S \end{matrix} \right\}$$

Ex (Toric Weak Nullstellensatz). $I \subseteq S$ homog. ideal. Then $\mathbb{V}_{X_\Sigma}(I) = \emptyset$ iff $B^l \subseteq I$ $\exists l > 0$

(multigrad.)

Def^n A virtual resolution is a min'l free ex with $H_i(F)$ B-torsion $\forall i > 0$

$\Leftrightarrow \sqrt{\operatorname{ann}(H_i(F))} \supseteq B$

↳ idea fixes issues w/ dim & length of MFRs for products of proj. spaces.

Notes - Multigraded Modules

Reference: CLS §5.2

Let $\mathbb{C}$ be an algebraically closed field

$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \mathbb{C}^*$ ← we write this because $\lambda \cdot \vec{x} = (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$ is an action of $\mathbb{C}^*$ on $\mathbb{A}^{n+1}$, then we quotient by orbits

$\pi : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$

Ideal-Variety Correspondance

$\{\text{Closed subvarieties of } \mathbb{P}^n\} \longleftrightarrow \{\text{homogeneous radical ideal } I \subseteq m = S_+ \subseteq S\}$

$$\begin{tikzcd} I & V(I) \arrow[l, mapsto] \\ I(V) & V \arrow[l, mapsto, leftarrow] \end{tikzcd}$$

Definition A minimal free resolution (graded) is a complex

$F_\bullet : F_0 \leftarrow F_1 \leftarrow \_\_ \leftarrow F_\ell \longleftarrow 0$ s.t. $F_i = \bigoplus S(j)^{\beta_{i,j}}$, $H_i(F_\bullet) = 0 \ \forall i > 0$

and $\text{im}(\partial_i) \subseteq m F_{i-1}$

Ex. $I = \langle x_1 x_3 - x_2^2, x_1 x_2 - x_0 x_3, x_0 x_2 - x_1^2 \rangle$

$$S \xleftarrow{(\text{gens})} S(-2)^3 \xleftarrow{\begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}} S(-3)^2 \longleftarrow 0$$

Theorem (Hilbert's Syzygy Theorem, 1890) ← Ddevelopped by Hilbert to prove existance of Hilbert formula

$X \subseteq \mathbb{P}^n$, $I(X) \subseteq S$ the ideal of X, then MFR of $S/I(X)$ has length $\leq n$

Free Resolutions give us: dimension, degree, etc

Eisenbud's Geometry of Syzygies

Big Question: Translate ↗ to multigradings

$S = \mathbb{C}[x_1, \ldots, x_N]$ a multigrading on S comes from an assignment of

$\deg(x_i) = a \in \mathbb{Z}^d$

Example $\mathbb{P}^1 \times \mathbb{P}^1$ has Cox Ring $S = \mathbb{C}[x_0, x_1, y_0, y_1]$ $\deg(x_i) = (1,0)$, $\deg(y_i) = (0,1)$

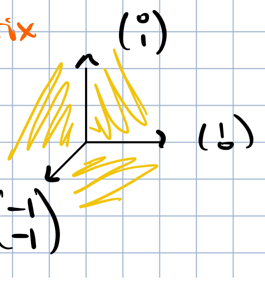
we want bihomogeneous equations

e.g $x_0^2 y_0 y_1 + x_0 x_1 y_1^2$

For $\mathbb{P}^2$

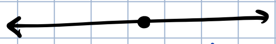
$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} \mathbb{Z}^3 \xrightarrow{[1\ 1\ 1]} \mathbb{Z} \longrightarrow 0$$

← degree matrix



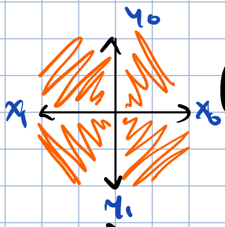
$\mathbb{P}^1$

$$0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix}-1\\1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{[1\ 1]} \mathbb{Z} \to 0$$



$\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\-1&0\\0&1\\0&-1\end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix}1&1&0&0\\0&0&1&1\end{bmatrix}} \mathbb{Z}^2 \to 0$$



$(\mathbb{C}^4 \setminus V(\langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle))/(\mathbb{C}^*)^2$

$(s,t)(x_0 : x_1 : y_0 : y_1) = (sx_0 : sx_1 : ty_0 : ty_1)$

Definition A fan Σ in $\mathbb{R}^n$ is a finite collection of cones (+adjectives) such that for all $\sigma \in \Sigma$ any face $\tau < \sigma$ is in Σ, and for any $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of each. ← like a simplicial complex

$\operatorname{supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma = |\Sigma|$ $\quad \Sigma(n) = n$ dim'l cones

For our case today: generated by linearly independent vectors (simplicial)

Let Σ be a simplicial fan, $N = \#\text{rays} = |\Sigma(1)|$ in $\mathbb{R}^d$

$$0 \longrightarrow \mathbb{Z}^d \xrightarrow{\substack{\text{matrix}\\ \text{of rays}\\ \text{of } \Sigma}} \mathbb{Z}^N \xrightarrow{[\text{grading on } S] = \deg} \mathbb{Z}^r \longrightarrow 0$$

$S = \mathbb{C}[x_1, \ldots, x_N]$

$B = \langle x^{\hat{\sigma}} := \prod_{i \notin \sigma} x_i \mid \sigma \in \Sigma \rangle$ ← max'l is enough

↑ irrelevant ideal

A simplicial toric variety X_Σ $(\mathbb{C}^n \setminus \operatorname{Var}(B))/(\mathbb{C}^*)^r$

Ex H_2

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\-1&2\\0&-1\end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix}1&-2&1&0\\0&1&0&1\end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

$S = \mathbb{C}[x_0, x_1, x_2, x_3]$

$\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}-2\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

$(\mathbb{C}^*)^2 \times \mathbb{C}^4 \longrightarrow \mathbb{C}^4$

$(s,t) \cdot (x_0, x_1, x_2, x_3) \longmapsto (sx_0, s^{-2}tx_1, sx_2, tx_3)$

$$\left\{\begin{array}{c}\text{Closed subvarieties of}\\ \text{simplicial toric varieties}\end{array}\right\} \begin{array}{c}\longrightarrow\\ \longleftarrow\end{array} \left\{\begin{array}{c}\text{homogeneous}\\ \text{radical ideals in } B\end{array}\right\}$$

Lecture #2

X a simplicial toric variety $\longleftrightarrow$ $S = \mathbb{C}[x_1, \text{——}, x_n]$ w/ $\mathbb{Z}^r$-grading
$\cup\!|$
B irrelevant ideal *monomial

Definition A virtual resolution is a minimal free complex w/ higher homology supported on B, ie annihilated by a power of B

Example $X = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{C}[x_0, x_1, y_0, y_1] \supseteq \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle$

$I_Y = \langle x_0, y_0 \rangle \cap \langle x_1, y_1 \rangle$ has MFR

$$\begin{tikzcd} S/I_Y & S \arrow[l] & S\binom{-2}{0} \oplus S\binom{-1}{-1}^2 \oplus S\binom{0}{-2} \arrow[l] & S\binom{-2}{-1}^2 \oplus S\binom{-1}{-2}^2 \arrow[l] & S\binom{-2}{-2} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

0 1 2 3 ← BAD

$S/(I_Y \cap B)$ has MFR *ask Christine how this is geometrically doing the same? → Is it unioning w/ empty set?

$$\begin{tikzcd} S/(I_Y \cap B) & S \arrow[l] & S\binom{-1}{-1}^2 \arrow[l] & S\binom{-2}{-2} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

*Koszul !

For $\mathbb{P}^m$ w/ $R = \mathbb{C}[\underline{x}]$

Recall $\operatorname{reg}(S/I) = \max\{j - i \mid \beta_{ij}(R/I) \neq 0\}$ = # rows of Betti table
Eisenbud-Goto $\Updownarrow$ ← truncate how?
$= \min\{\ell \mid [R/I]_{\geq \ell} \text{ has a linear resolution}\}$
$= \max\{i + j \mid [H^i_{(\underline{x})}(R/I)]_j \neq 0\}$ ← local cohomology

Definition (Maclagan-Smith, 2005) $d \in \mathbb{Z}^r$, assume $H^0_B(S/I) = 0$

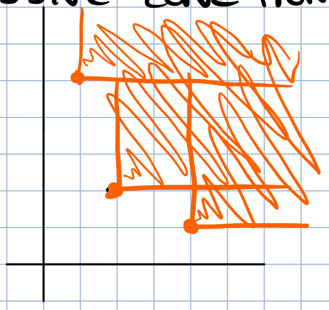
Then S/I is d-regular if $\forall\, i > 0$, $u \in \deg(S)$ w/ $\sum_j u_j = i - 1$, $H^i_B(S/I)_{d-u} = 0$

Notation $\operatorname{reg}(S/I) := \{d \in \mathbb{Z}^r \mid S/I \text{ is } d\text{-regular}\} \subseteq \mathbb{Z}^r$

Examples Y = 2 points in $\mathbb{P}^1 \times \mathbb{P}^1$

$\operatorname{reg}(S/I_Y) = \left[\binom{1}{0} + \mathbb{N}^2\right] \cup \left[\binom{0}{1} + \mathbb{N}^2\right]$

Massive curve from exercizes: (in $\mathbb{P}^1 \times \mathbb{P}^2$)



*Marc Chardin + students extracting regularity from free resolution

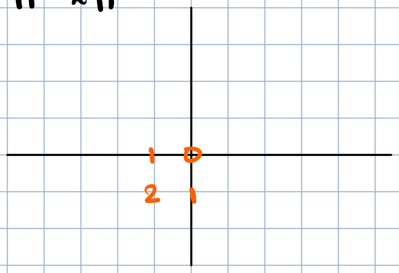
Theorem (B.-Erman-Smith, 2019) $X = \mathbb{P}^{n_1} \times \_\_ \times \mathbb{P}^{n_r}$

$Y \subseteq X$, $\operatorname{dereg}(S/I_Y)$, $F_\bullet$ = MFR of S/I_Y, then the subcomplex of $F_\bullet$ w/ summands generated in degrees $\leq d+n$, w/ $n = (n_1, \_\_, n_r)$ is a virtual resolution

Analogue of Linearity

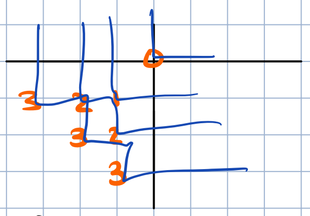
Koszul Complex on variables for $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{tikzcd} S & S\binom{-1}{0}^2 \oplus S\binom{0}{-1}^2 \arrow[l] & S\binom{-1}{-1}^3 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$



What makes a res Hilbert-Burch?
minors of matrix give entries in the next one

For S/B



What are these pictures??

Virtual Eisenbud-Goto Linearity [Bruce - Cranton Heller - Sayrafi]

$Y \subseteq \mathbb{P}^{n_1} \times \_\_ \times \mathbb{P}^{n_r}$

shift diagram so 0 at $-d$ and look @ where Betti #'s occur

$\operatorname{dereg}(Y) \iff [S/I_Y]_{\geq d}$ has a quasilinear mfr

I am so lost on pictures!

Packages

Normal Toric Varieties

Virtual Resolutions

Truncations

Lecture #3

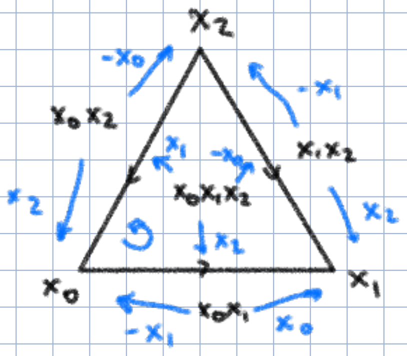
X a simplicial toric variety, $S = \mathrm{Cox}(S)$ is $\mathbb{Z}^r$-graded

← using Mirror Symmetry + symplectic geometry

Theorem (Hanlon-Hicks-Lazarev, 2024) $Y \hookrightarrow X$ a normal subvariety, there exists a virtual resolution of S/I_Y of length $\mathrm{codim}_X(Y)$

Corollary Virtual Hilbert Syzygy theorem

Example Taylor resolution $F = S^3$, $S = \mathbb{C}[x_0, x_1, x_2] \leftrightsquigarrow \mathbb{P}^2$



$$\begin{tikzcd}
\Lambda^0 F \arrow[d, equal] & \Lambda^1 F \arrow[l, "(x_0\ x_1\ x_2)"'] \arrow[d, equal] & & \Lambda^2 F \arrow[ll, "\begin{bmatrix} -x_1 & x_2 & 0 \\ x_0 & 0 & x_2 \\ 0 & -x_0 & -x_1 \end{bmatrix}"'] \arrow[d, equal] & \Lambda^3 F \arrow[l, "\begin{bmatrix} x_2 \\ x_1 \\ -x_0 \end{bmatrix}"'] \arrow[d, equal] & 0 \arrow[l] \\
S & S(-1)^3 & & S(-2)^3 & S(-3) &
\end{tikzcd}$$

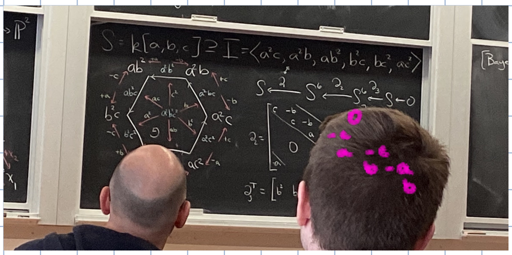
- * X is an oriented regular cell complex w/ ε (signs)
- * a monomial label m_σ on $\sigma \in X$ such that $m_{\sigma'} \mid m_\sigma$ $\forall\ \sigma' \subseteq \sigma$

Definition A cellular complex supported on X for $\{m_\sigma \mid \sigma \in X\}$

$$F_X := \bigoplus_{\sigma \in X} S_\sigma \quad \text{with } \partial_\sigma = \sum_{\sigma' \in X} \varepsilon(\sigma, \sigma') \frac{m_\sigma}{m_{\sigma'}} \sigma'$$

& homological degree $\dim \sigma + 1$

Theorem F_X is a free resolution iff $X_{\leq b}$ is acyclic $\forall\ b \in \mathbb{Z}^n$



$\partial_j^T = [b^2\ bc\ c^2\ ac\ a^2\ ab]$

Example $p = [1:1:1] \in \mathbb{P}^2$ $\mathrm{Cox}(P) = \mathbb{C}[x_0, x_1, x_2]$ w/ standard grading

$I_p = \langle x_1 - x_0, x_2 - x_0 \rangle$

$$\begin{tikzcd}
S & S(-1)^2 \arrow[l, "{[x_1 - x_0\ \ x_2 - x_0]}"'] & & S(-2) \arrow[ll, "\begin{bmatrix} x_2 - x_0 \\ x_1 - x_0 \end{bmatrix}"'] & 0 \arrow[l]
\end{tikzcd}$$

Example $C = \mathrm{im}(\mathbb{P}^1 \hookrightarrow \mathbb{P}^3)$ $[x_0 : x_1] \longmapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$

$I_C = \langle x_0x_3 - x_1x_2,\ x_1^2 - x_0x_2,\ x_2^2 - x_1x_3 \rangle$

Lattice ideal $L \subseteq \mathbb{Z}^n$, $I_L = \langle x^a - x^b \mid a - b \in L \rangle \longleftrightarrow Y$

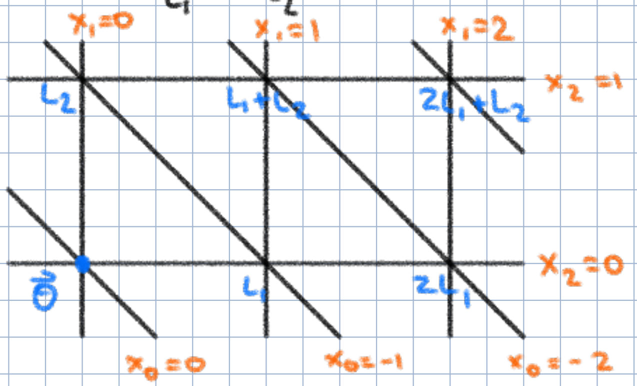
toric ideal $\longleftrightarrow$ irreducible

$L = \ker[1\ 1\ 1] = \mathbb{Z}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$

$L = $ kernel of $\begin{bmatrix}3 & 2 & 1 & 0\\ 0 & 1 & 2 & 3\end{bmatrix}$

$$L = \mathbb{Z}\cdot\left\{\underset{L_1}{\begin{pmatrix}-1\\1\\0\end{pmatrix}}, \underset{L_2}{\begin{pmatrix}-1\\0\\1\end{pmatrix}}\right\} \longrightarrow \mathbb{R}L \cap \{x_i = j\}_{\substack{0 \le i \le 2\\ j \in \mathbb{Z}}}$$

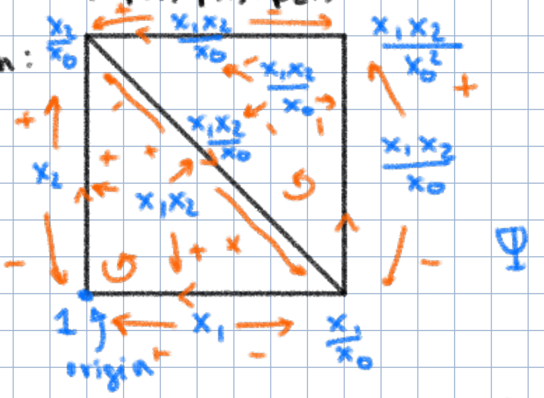
$\longrightarrow$ hidden conditions



$\chi : \mathbb{R}L \longrightarrow \mathbb{Z}^3$

$p \longmapsto (\lceil p_0 \rceil, \lceil p_1 \rceil, \lceil p_2 \rceil)$

Zoom in:



$\chi : \mathbb{R}L \longrightarrow \mathbb{Z}^n$

$p \longmapsto \chi(p) \rightsquigarrow$ monomial $m_{\chi(p)} \in T \leftarrow$ Laurent polynomials

Definition cellular complex supported on Ψ for $\{m_{\chi(\sigma)} \mid \sigma \in \Psi\}$

$F_\chi : \bigoplus_{\sigma \in \Psi} S_\sigma$ w/ same differential

* Can still use Bayer-Sturmfels result w/ $X_{\le \chi(b)}$ $X = \mathbb{R}L \cap \{x_i = j\}$

$$\begin{array}{l} S[L] = S[z^a \mid a \in L] \subseteq S[z_1^{\pm 1}, \_\_\_, z_n^{\pm 1}] \\ \quad\downarrow \\ S\ \ = S[L] / \langle z^a - 1 \mid a \in L \rangle \\ \cup\!\!\mid \\ I_L \end{array}$$

Theorem [B-Cranton Heller-Smith-Yang] $F_\chi \otimes_S S[L] = HHL$ (up to sheafification) $\leftarrow$ à la 3.1 in BS

and resolves $\overline{S/I_Y}$

6.3 Tutorial Problems

Here are the associated tutorial problems.

Let $\Bbbk$ be an algebraically closed field. The `NormalToricVarieties` package in `Macaulay2` will be helpful for the later problems.

Regular:

1. What is the Cox ring, fan, and fundamental short exact sequence for

$$\Bbbk^1 \times \mathbb{P}^1? \qquad \Bbbk^2? \qquad \mathbb{P}^1 \times \mathbb{P}^2? \qquad \mathbb{P}^n?$$

Hint: Work backwards from the fundamental short exact sequence.

2. Let $X = \mathbb{P}^1 \times \mathbb{P}^2$ with $S = \mathrm{Cox}(X) = \mathbb{C}[x_0, x_1, y_0, y_1, y_2]$.
   (a) Find a radical ideal I_p whose variety is precisely the point $p = ([1:2],[1:3:4])$.
   (b) Find a radical ideal for the point $([a:b],[c:d:e])$ for any choice of $a, b, c, d, e \in \Bbbk$ with $a \neq 0$ and $c \neq 0$.

3. (a) The Weak Nullstellensatz for $\Bbbk^n$ says that for any ideal $J \subseteq \Bbbk[x_1, x_2, \ldots, x_n] = S$, $I(\mathrm{Var}_{\Bbbk^n}(J)) = \sqrt{J}$, where

$$\sqrt{J} = \{f \in S \mid \exists \ell \geq 0 : f^\ell \in J\} \text{ is the \textit{radical} of } J,$$

   which is again an ideal of S. Use this to show that for ideals I and B in S, $\mathrm{Var}_{\Bbbk^n}(I) \subseteq \mathrm{Var}_{\Bbbk^n}(B)$ is equivalent to the existence of some $\ell \geq 0$ such that $B^\ell \subseteq I$.
   (b) Prove the Toric Weak Nullstellensatz: Let X_Σ be a simplicial toric variety with total coordinate ring S and irrelevant ideal B. If $I \subseteq S$ is a homogeneous ideal, then

$$\mathrm{Var}_X(I) = \varnothing \text{ in } X_\Sigma \iff B^\ell \subseteq I \text{ for some } \ell \geq 0.$$

4. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Consider $I = \langle x_0, y_0 \rangle \cap \langle x_1, y_1 \rangle \subseteq S = \mathrm{Cox}(X) = \Bbbk[x_0, x_1, y_0, y_1]$ with irrelevant ideal $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$.
   (a) What is the variety $\mathrm{Var}_X(I)$ of I inside X?
   (b) Compute graded minimal free resolutions of S/I, $S/I \cap \langle x_0, x_1 \rangle$, $S/I \cap \langle y_0, y_1 \rangle$, and $S/I \cap B$, and compare their lengths.

5. Let $S = \mathrm{Cox}(\mathbb{P}^1 \times \mathbb{P}^2) = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ with irrelevant ideal $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$. The curve from lecture is defined by the ideal

$$I_C = \langle x_0^2 y_0^2 + x_1^2 y_1^2 + x_0 x_1 y_2^2, x_0^3 y_2 + x_1^3 (y_0 + y_1)) \rangle : B^\infty.$$

   In Macaulay2, compute the multigraded resolution for S/I_C. Which powers of (components of) B can you intersect with I_C to construct shorter virtual resolutions for S/I_C?

6. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with $S = \mathrm{Cox}(X) = \mathbb{C}[x_0, x_1, y_0, y_1]$. Consider the set of five points

$$Y = \{([1:1],[1:1]),\ ([1:2],[1:2]),\ ([1:3],[1:3]),\ ([1:4],[1:4]),\ ([1:6],[1:8])\}.$$

   In Macaulay2, compute the B-saturated radical ideal I_Y defining Y. Then find the graded minimal free resolution of $S/(I_Y \cap \langle x_0, x_1 \rangle^a)$ for $0 \leq a < 5$.

Advanced:

7. Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ be a product of projective spaces. Fix coordinates for a point $p = (p_1, p_2, \dots, p_r)$, where $p_i \in \mathbb{P}^{n_i}$, and write down a defining ideal for p inside $S = \mathrm{Cox}(X)$.

8. Consider $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ with Cox ring S and irrelevant ideal
$$B = \bigcap_{i=1}^{r} \langle x_{i,j} \mid 0 \leq j \leq n_i \rangle.$$
Let $Z \subseteq X$ be a finite collection of points. For small values of $n = (n_1, n_2, \dots, n_r)$, experiment in Macaulay2 to find tuples $a = (a_1, a_2, \dots, a_r)$ such that $S/I \cap B^a$ has a resolution of length $|n| = n_1 + n_2 + \cdots + n_r$. (Here, $B^a = \bigcap_{i=1}^{r} \langle x_{i,j} \mid 0 \leq j \leq n_i \rangle^{a_i}$.)

9. Let Δ be a simplicial complex and consider I_Δ inside the Cox ring S of some simplicial toric variety X. How can you modify Δ to Δ' so that the Stanley–Reisner ideals I_Δ and $I_{\Delta'}$ define the same subvariety of X?

10. Let X be a simplicial toric variety with $S = \mathrm{Cox}(S)$ and irrelevant ideal B, so that there is a quotient map $\pi\colon (\Bbbk^n \setminus \mathrm{Var}_{\Bbbk^n}(B)) \to X$. Assume that given a homogeneous ideal $I \subseteq S$, then
$$\mathrm{Var}_X(I) = \{\pi(x) \in X \mid f(x) = 0 \text{ for all } I\}$$
is a closed subvariety of X and all subvarieties of X arise in this way.
    (a) Show that there is a bijective correspondence
$$\{\text{closed subvarieties of } X\} \leftrightarrow \left\{\begin{matrix}\text{homogeneous radical} \\ \text{ideals } I \subseteq B \subseteq S\end{matrix}\right\}.$$
    (b) Show that there is a bijective correspondence
$$\{\text{homogeneous radical ideals } I \subseteq B \subseteq S\} \leftrightarrow \left\{\begin{matrix}\text{homogeneous } B\text{-saturated} \\ \text{radical ideals } I \subseteq S\end{matrix}\right\}.$$

Let $\Bbbk$ be an algebraically closed field.

Regular:

1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Consider
$$I = \langle x_0, y_0 \rangle \cap \langle x_1, y_1 \rangle \subseteq S = \operatorname{Cox}(X) = \Bbbk[x_0, x_1, y_0, y_1]$$
with irrelevant ideal $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$.
   (a) Use Macaulay2 to compute the multigraded regularity of S/I.
   (b) Compute the resolution of a pair for each generator of the regularity.
   (c) Compute the resolution of $[S/I]_{\geq d}$ for each generator of the regularity and compare them with the quasilinearity conditions.

2. Let $S = \operatorname{Cox}(\mathbb{P}^1 \times \mathbb{P}^2) = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ with irrelevant ideal $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$. The curve from lecture is defined by the ideal
$$I_C = \langle x_0^2 y_0^2 + x_1^2 y_1^2 + x_0 x_1 y_2^2, x_0^3 y_2 + x_1^3 (y_0 + y_1)) \rangle : B^\infty.$$
   (a) Use Macaulay2 to compute the multigraded regularity of S/I.
   (b) Compute the resolution of a pair for each generator of the regularity.
   (c) Compute the resolution of $[S/I_C]_{\geq d}$ for each generator of the regularity and compare them with the quasilinearity conditions.

3. Use the Bayer–Sturmfels construction to compute a cellular free resolution for
$$I = \langle a^2 b, ac, bc^2, b^2 \rangle \subseteq \Bbbk[a, b, c].$$
   (a) Use the simplicial complex that is two triangles glued along one side, where the monomials of degree two to label the vertices of the glued edge.
   (b) Why does the opposite choice of diagonal not produce a free resolution?

4. (a) Compute a cellular free resolution for the irrelevant ideal B when $X = \mathbb{P}^n$.
   (b) Compute a cellular free resolution for the irrelevant ideal B when $X = \mathbb{P}^1 \times \mathbb{P}^n$. You can use a polytope as the underlying space for the cellular resolution of B when X is any simplicial toric variety.
   (c) Relate the polytope from (b) in terms of the fan for X? Does this same idea work for any simplicial toric variety? (It should!)

Advanced:

5. Let X be the second Hirzebruch surface, with rays $\{(1,0),(0,1),(-1,2),(0,-1)\}$.
   (a) Plot the images of the coordinate hyperplanes in $\mathbb{Z}^4$ inside $\mathbb{R}L \cong \mathbb{R}^2$, where
   $$L = \operatorname{span}_{\mathbb{Z}}\{(1,0,-1,0),(0,1,2,-1)\}.$$
   (b) With L as in (a), use the function use the function
   $$\psi\colon \mathbb{R}L \to \mathbb{Z}^4 \quad \text{with} \quad a \mapsto \lceil a \rceil$$
   to produce a cellular resolution of S/I_p, where p is the identity point in X, where $S = \operatorname{Cox}(X)$.
   (c) With L as in (a), use the function
   $$\psi\colon \mathbb{R}L \to \mathbb{Z}^8 \quad \text{with} \quad a \mapsto (\lceil a \rceil, -\lfloor a \rfloor)$$
   to produce a cellular resolution of R/I_Δ, where Δ is the image of $X \hookrightarrow X \times X$ given by $x \mapsto (x,x)$ and $R = \operatorname{Cox}(X \times X)$.
   (d) Repeat part (c) with the function
   $$\psi\colon \mathbb{R}L \to \mathbb{Z}^8 \quad \text{with} \quad a \mapsto (\lfloor a \rfloor, -\lfloor a \rfloor)$$
   to produce a cellular resolution of R/I_Δ. (In general, this function only yields a virtual resolution, but this one happens to be acyclic.)

7. Gröbner Geometry and Applications (S. Da Silva and P. Klein)

This course explored the use of Gröbner bases, degenerations, and related combinatorics to study problems in commutative algebra and algebraic geometry. This course was taught by Sergio Da Silva (Virginia State) and Patricia Klein (Texas A&M).

7.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

7.2 Lecture Notes

We have included copies of Sergio and Patricia's lecture notes and their tutorials.

Lecture 1

Let $\mathbb{K}$ be any field, $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ an ideal, and $<$ a monomial order on R.

Recall that $\text{init}_<(I) = \langle \text{init}_<(f) \mid f \in I \rangle$ and $g_1, \dots, g_s \in R$ is a Gröbner basis of I (with respect to $<$) if $\text{init}_<(I) = \langle \text{init}_<(g_i) \mid 1 \le i \le s \rangle$.

we called this LT(I) last week →

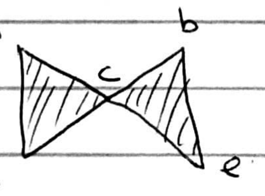
Approach: Use Gröbner degeneration to study properties of R/I using $R/\text{init}_<(I)$, properties like reducedness, being Cohen-Macaulay, being Gorenstein, normality, etc.

See Eisenbud chapter 15 →

Exercise: If $\text{init}_<(I)$ is squarefree, then I is radical.

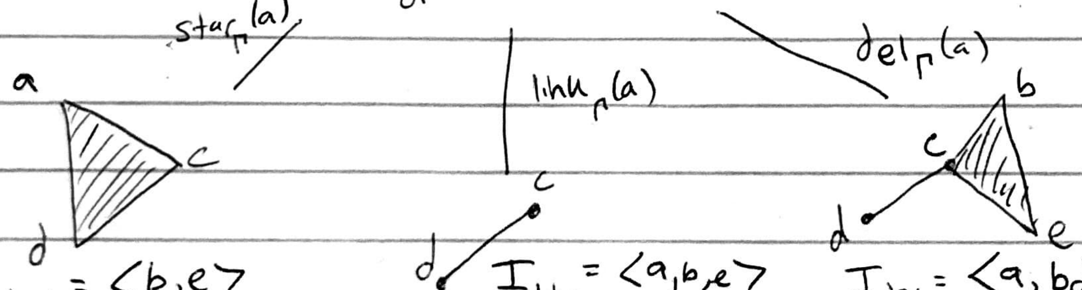
A simplicial complex Γ is vertex decomposable if
- Γ is pure and,
- $\Gamma = \emptyset$ or a simplex or,
- $\exists$ v such that both $\text{link}_\Gamma(v)$ and $\text{del}_\Gamma(v)$ are vertex decomposable.

Example: Γ:



$I_\Gamma = \langle ab, ae, bd, de \rangle$

As complexes on vertizes a,b,c,d,e →



$I_{\text{star}} = \langle b, e \rangle$ $I_{\text{link}} = \langle a, b, e \rangle$ $I_{\text{del}} = \langle a, bd, de \rangle$

Choosing $v=a$ does not yield a vertex decomposition.

Lemma: $I_\Gamma = I_{\text{star}(v)} \cap I_{\text{del}(v)}$

However, viewed as complexes in the vertices b, c, d, e:

$I_{\text{link}(a)} = \langle b, e \rangle$, $I_{\text{del}(a)} = \langle bd, de \rangle$ and $I_\Gamma = I_{\text{link}(a)} \cap (I_{\text{del}(a)} + \langle a \rangle)$

Notice that if $I_\Gamma = \langle x_v q_1, \ldots, x_v q_k, h_1, \ldots, h_\ell \rangle$ with $x_v \nmid q_i, h_i$ for any i, then

$I_{\text{link}(v)} = \langle q_1, \ldots, q_k, h_1, \ldots, h_\ell \rangle$ $I_{\text{del}(v)} = \langle h_1, \ldots, h_\ell \rangle$

Geometric vertex decomposition allows us to generalize this construction for any ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. To do this, assume that $<$ is a lex order on R with $y = x_j$ for some j the greatest variable. Write the Gröbner basis for I as

$$y^{d_1} q_1 + r_1, \ldots, y^{d_k} q_k + r_k, h_1, \ldots, h_\ell$$

where $d_i > 0$, y does not divide any term of q_i or h_i, and $\text{init}_y(y^{d_i} q_i + r_i) = y^{d_i} q_i$

Example: $\text{init}_y(x^2 y + zy + \underbrace{w^3}_{r_i}) = \underbrace{(x^2 + z)}_{q_i} y$ $\leftarrow d_i = 1$

$\text{init}_y(z + xw) = z + xw$

Set $C_{y,I} = \langle q_1,\dots,q_k,h_1,\dots,h_\ell\rangle$ $N_{y,I} = \langle h_1,\dots,h_\ell\rangle$

Exercise: Show $C_{y,I} = (\text{init}_y(I) : y^\infty)$ and
$N_{y,I} + \langle y\rangle = \text{init}_y(I) + \langle y\rangle$

I is geometrically vertex decomposable (GVD) if
I is unmixed with

$$\text{init}_y(I) = \langle y^{d_1}q_1,\dots,y^{d_k}q_k,h_1,\dots,h_\ell\rangle = C_{y,I}\cap(N_{y,I}+\langle y\rangle)$$

and the contraction of $C_{y,I}$ and $N_{y,I}$ to
$\mathbb{K}[x_1,\dots,\hat{y},\dots,x_n]$ is GVD.

Base Cases: $\langle 0\rangle, \langle 1\rangle$ or $\langle x_{i_1},\dots,x_{i_r}\rangle$, $1\le r\le n$

Example: $I = \langle ab-c^2\rangle \subset \mathbb{K}[a,b,c]$

For the lex order $a>b>c$:

$N_{a,I}=\langle 0\rangle$, $C_{a,I}=\langle b\rangle$, $\text{init}_a(I)=\langle ab\rangle = \underbrace{\langle b\rangle}_{C_{a,I}}\cap(\underbrace{\langle a\rangle}_{N_{a,I}}+\langle 0\rangle)$

For the lex order $c>a>b$:
$N_{c,I}=\langle 0\rangle$, $C_{c,I}=\langle 1\rangle$ and $\text{init}_y(I)=\langle c^2\rangle \neq C_{c,I}\cap(N_{c,I}+\langle c\rangle)$

See exercises → **Theorem**: $\text{init}_y(I) = C_{y,I}\cap(N_{y,I}+\langle y\rangle)$ iff I is squarefree in y

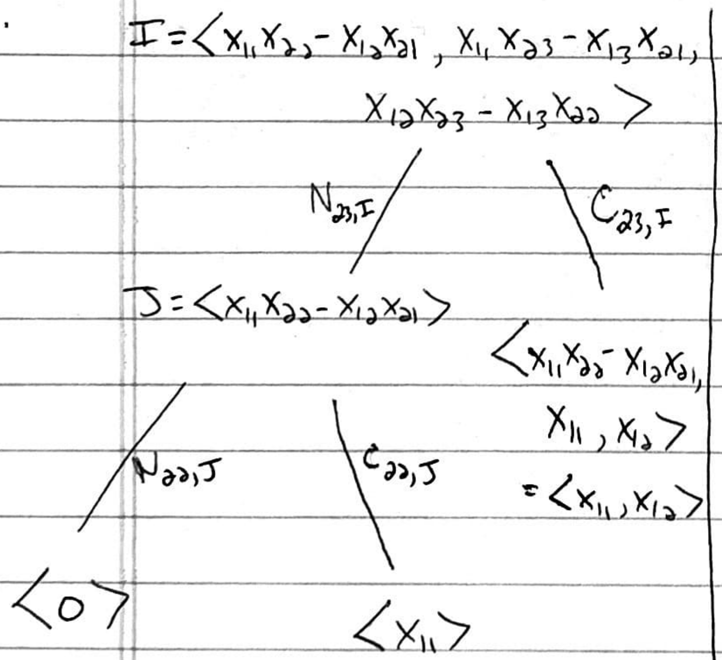
Theorem: If I is GVD, then it is radical. If it is also homogeneous, then R/I is Cohen-Macaulay.

Lex-compatible GVDs

Consider the ideal I generated by all 2×2 minors of $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}$.

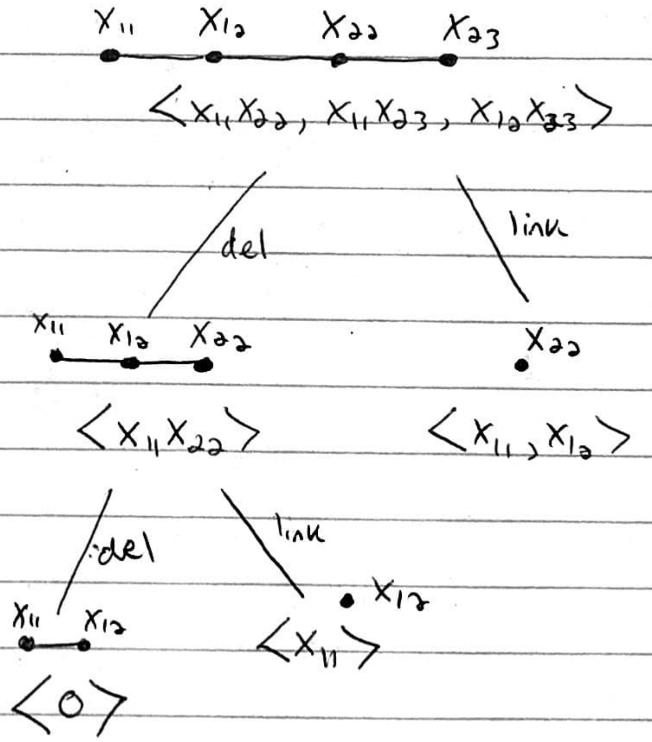
Using the lex order $x_{23} > x_{22} > \dots > x_{11}$, we get $\mathrm{init}_<(I) = \langle x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23} \rangle$

From last week, we saw that $\mathrm{init}_<(I) = I_\Gamma$ with Γ vertex decomposable.



I is homogeneous and GVD
$\Rightarrow R/I$ CM and I radical

(Remark 1: we also get glicci)
(Remark 2: We can change order at each step)



Γ vertex decomposable $\Rightarrow R/I_\Gamma$ CM
I_Γ squarefree $\Rightarrow I_\Gamma$ radical

Both properties must be true for I via Gröbner degeneration

Theorem : I_Γ is GVD iff Γ is vertex decomposable

Theorem : $ht(I)$, $reg(R/I)$, $e(R/I)$, $a(R/I)$ can all be computed from C and N.

Non-degenerate case : $ht(I) = ht(C) = ht(N) + 1$

$$\left\{\begin{array}{l} reg(R/I) = \max\{reg(R/N), reg(R/C)+1\} \\ e(R/I) = e(R/N) + e(R/C) \\ a(R/I) = \max\{a(R/N), a(R/C)\} + 1 \end{array}\right.$$

See "Three invariants of geometrically vertex decomposable ideals"

Mike + Adam have M2 code GeometricDecomposability

Patricia Klein
June 11, 2025 +
June 13, 2025

Gröbner Geometry: Lectures 2 and 3

Part 1: Gröbner degeneration (Classical setting): Roughly, we want to replace a variety V by a "union" of coordinate subspaces that remembers key information about V (e.g. dimension, degree).

Show Desmos: $sxy - tz^2 = 0$, slide $0 \xleftrightarrow{t} 1$ $\quad 0 \xleftrightarrow{s} 1$

$u\,y(x-z) - z^2 = 0$, slide $1 \xleftrightarrow{u} 500$

$x^2 + y^2 + vz^2 = 1$, slide $1 \xleftrightarrow{v} 100$

Recall: R is a standard graded polynomial ring. I, J, etc. are homog. ideals

- $\mathrm{Hilb}_{R/I} = \mathrm{Hilb}_{R/\mathrm{in}_< I}$ $\forall$ term orders $<$
  (Exercise: If $J \subseteq \mathrm{in}_< I$, then $J = \mathrm{in}_< I$ iff $\mathrm{Hilb}_{R/I} = \mathrm{Hilb}_{R/J}$)
- $\dim(R/I) = (\text{degree Hilb. polynomial}) + 1$
  $(\Rightarrow \mathrm{ht}(I) = \mathrm{ht}(\mathrm{in}_<(I)) = \mathrm{ht}(\mathrm{in}_y(I))$, Ex. 2)
- degree = normalized leading coefficient of Hilb. polynomial

Fact: · If $\mathrm{in}_< I = \sqrt{\mathrm{in}_< I}$, then

$\deg(\mathbb{V}(I)) = \deg(\mathbb{V}(\mathrm{in}_< I)) = \#$ top dim'l components

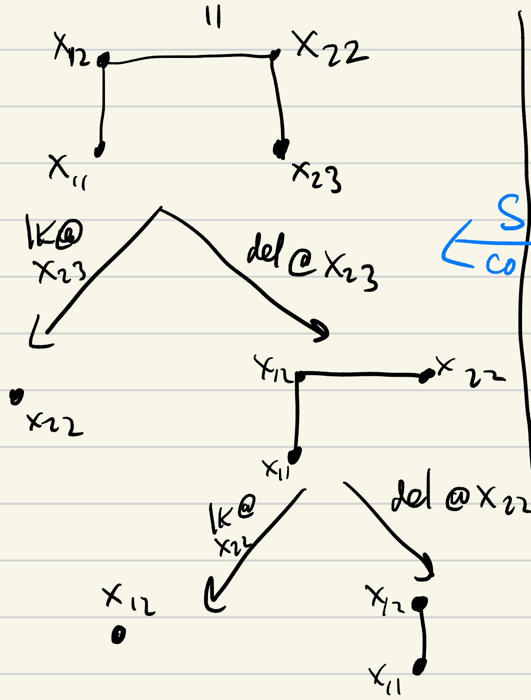
Recall from Lecture 1:

Let $<$ be the lexicographic order on $x_{11}<x_{12}<x_{13}<x_{21}<x_{22}<x_{23}$.

Let $M_3 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$, $I_2(M_3) = (\underbrace{x_{11}x_{23}}_{\text{wt } 1} - \underbrace{x_{21}x_{13}}_{\text{wt } 0}, \underbrace{x_{12}x_{23}}_{\text{wt } 1} - \underbrace{x_{22}x_{13}}_{\text{wt } 0}, \underbrace{x_{11}x_{22} - x_{12}x_{21}}_{\text{wt } 0})$

$\Delta(\text{in}_<(I_2(M_3)))$

$\|$

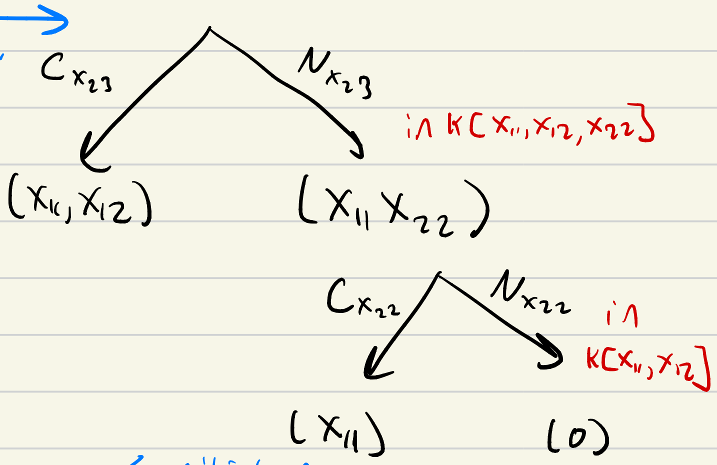


SR corr.

$(x_{11}x_{23}, x_{12}x_{23}, x_{11}x_{22})$

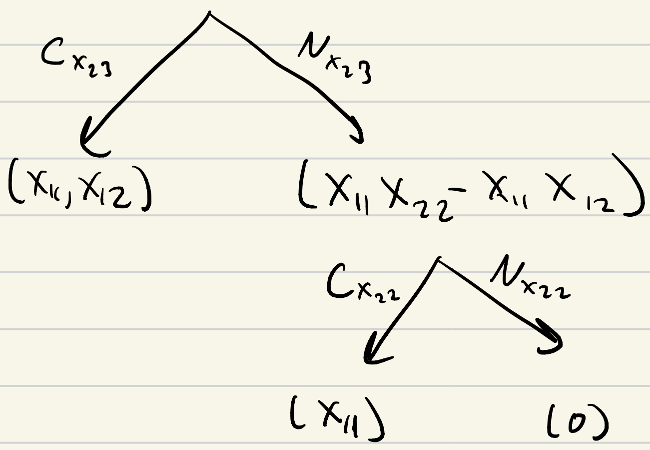
$\|$

$\text{in}_<(I_2(M_3))$ in $K[x_{11},x_{12},x_{22},x_{23}]$



$<$-initial

$I_2(M_3)$



Part 2: Goal of lecture: Show that the ideal generated by the size 2 minors of a generic $2 \times n$ matrix is gvd. State some consequences. Let $I_2(\text{matrix}) = \langle \text{size 2 minors of matrix} \rangle$

$$M_n = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{bmatrix}. \qquad I = I_2(M_n) = (x_{1i}x_{2j} - x_{1j}x_{2i} \mid 1 \le i < j \le n),$$

(☆) <u>Prop</u>: If C, N homogeneous, y homog. Set $J = yC + N$.
If $N : y = N$, N CM, $\mathrm{ht}\, N - \mathrm{ht}\, C + 1$, then

- $\mathrm{ht}\, J = \mathrm{ht}\, C$
- J CM (unmixed, resp.) $\Longleftrightarrow$ C CM (unmixed, resp.)

<u>Proof</u> ingrediat: Local cohomology (See Migliore-Nagel, "Notes from Torino summer school")
<u>Key word</u>: "Basic double G-link" (when N is G_0)

(*) <u>Prop</u>: · If $\mathrm{in}(I)$ is CM, then I is CM. · $\mathrm{ht}(\mathrm{in}_y(I)) = \mathrm{ht}(I)$
<u>Proof</u> <u>ingrediat</u>: Free resolutions, limits.

Assume by induction that

- $\mathrm{ht}\ I_2(M_{n'}) = n' - 1 \quad \forall\ 2 < n' < n$
- $I_2(M_{n'})$ is gvd ($\Rightarrow$ CM)

Base case $n' = 2$: · $\mathrm{ht}(x_{11}x_{22} - x_{12}x_{21}) = 1 = 2 - 1$ ✓
· $C(x_{22}, x_{11}x_{22} - x_{12}x_{21}) = (x_{11})$, $N(x_{22}, x_{11}x_{22} - x_{12}x_{21}) = 0$ ✓

<u>Fact</u>: The maximal minors of a generic matrix form a universal Gröbner basis (i.e. a Gröbner basis under any term order).

$$\mathrm{in}_{x_{2n}}(I) = \underset{\text{ht } n-1,\ CM}{\overset{\overset{C_{x_{2n},I}}{\shortparallel}}{x_{2n}(x_{11},\dots,x_{1n-1})}} + \underset{\text{ht } n-2,\ CM}{\overset{\overset{N_{x_{2n},I}+(x_{2n})}{\shortparallel}}{I_2(M_{n-1})}} = \underset{\substack{\text{GVD}\\ \text{by}\\ \text{defn.}}}{\overset{\overset{C_{x_{2n},I}}{\shortparallel}}{(x_{11},\dots,x_{1n-1})}} \cap \underset{\substack{\text{GVD}\\ \text{by}\\ \text{induction}}}{\overset{\overset{N_{x_{2n},I}+(x_{2n})}{\shortparallel}}{(I_2(M_{n-1}), x_{2n})}}$$

Conclusions: ① $\mathrm{in}_<(I)$ is unmixed of ht $n-1$ by $\bigstar$

② $\mathrm{in}_<(I)$ is GVD. $R/\mathrm{in}_<(I)$ is CM.

③ R/I is CM by ②, hence unmixed. Now GVD by defn.

④ R/I is regular in codim 1 by special case of Ex. 12

⑤ R/I is normal by ③+④

⑥ I is prime by Ex. 10

//

We can also use geometric vertex decomposition to establish Gröbner bases, in the mold of Migliore-Nagel-Gorla '13.

Prop (K.-Rajchgot): Let $I=(yq_1+r_1,\dots,yq_k+r_k,h_1,\dots,h_e)$ be a homogeneous ideal of R with $y=x_i$, $y\nmid$ any term of q_i, r_i, h_i. Fix a term order $<$, and suppose that

$$\mathcal{G}_C=\{q_1,\dots,q_k,h_1,\dots,h_e\} \text{ and}$$
$$\mathcal{G}_N=\{h_1,\dots,h_e\} \text{ are Gröbner bases for}$$

the ideals they generate, which we call C and N, respectively. Assume that $\mathrm{in}_<(yq_i+r_i)=y\cdot\mathrm{in}_<(q_i)$ $\forall i$. Assume also that $\mathrm{ht}(I),\mathrm{ht}(C)>\mathrm{ht}(N)$ and that N is unmixed. If

$$I_2\begin{pmatrix} q_1 & \cdots & q_k \\ r_1 & \cdots & r_k \end{pmatrix}\subseteq N,$$

then the given generators of I are a Gröbner basis.

ht 1, prime

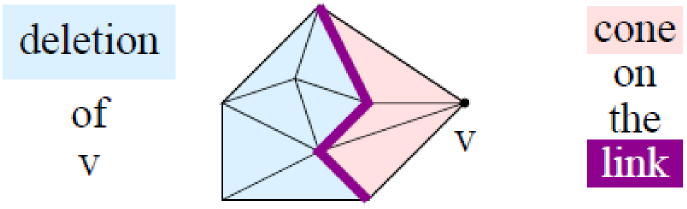
Ex: $R=K[x_{11},\dots,x_{23}]$, $y=x_{23}$. $N=(x_{11}x_{22}-x_{12}x_{21})$, $C=(x_{11},x_{12})$

$I=(\underbrace{x_{23}}_{y}\underbrace{x_{11}}_{q_1}+\underbrace{-x_{21}x_{13}}_{r_1},\ \underbrace{x_{23}}_{y}\underbrace{x_{12}}_{q_2}+\underbrace{-x_{22}x_{13}}_{r_2},\ x_{11}x_{12}-x_{12}x_{21})$

$$q_2r_1-q_1r_2\left(=q_2(yq_1+r_1)-q_1(yq_2+r_2)\right)=x_{12}(-x_{21}x_{13})-x_{11}(-x_{22}x_{13})$$
$$=x_{13}(x_{11}x_{22}\underset{\substack{\uparrow\\ N}}{-}x_{12}x_{21})$$

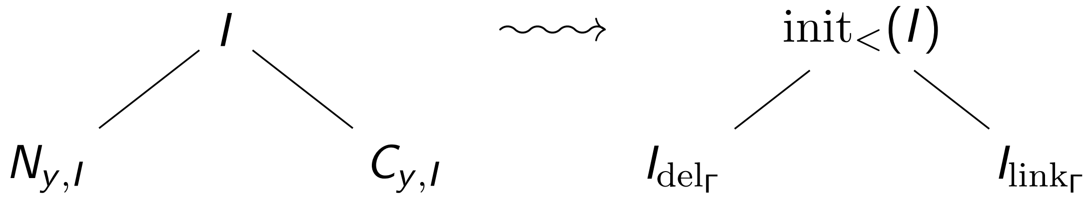
//

Lecture 3b: Examples of GVD Ideals



Gröbner Geometry and Applications
June 13, 2025

Geometric Vertex Decomposition



I

$N_{y,I}$ $C_{y,I}$

$\rightsquigarrow$

$\mathrm{init}_<(I)$

I_{del_Γ} I_{link_Γ}

- A squarefree monomial ideal I_Γ is GVD iff Γ is vertex decomposable.
- I GVD $\Rightarrow$ I is radical.
- I homogeneous and GVD $\Rightarrow$ then R/I is Cohen-Macaulay and I glicci.

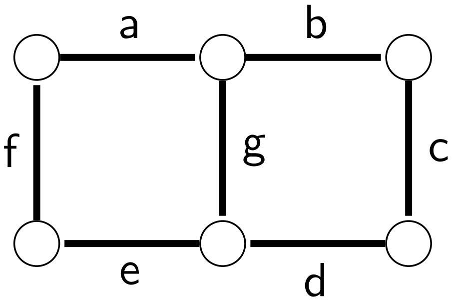
Toric Ideals of Graphs

- Let $G = (V(G), E(G))$ be a finite simple graph.
- Define $\mathbb{K}[E(G)] = \mathbb{K}[e_1, \ldots, e_n]$ and $\mathbb{K}[V(G)] = \mathbb{K}[v_1, \ldots, v_r]$.
- Consider the $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[E(G)] \to \mathbb{K}[V(G)]$

$$\varphi(e_i) = x_j x_k \;\text{ where } e_i = \{x_j, x_k\} \text{ for all } i \in \{1, \ldots, n\}.$$

The *toric ideal* I_G *of the graph* G is the kernel of φ and is generated by binomials corresponding to closed even walks on G.

Properties

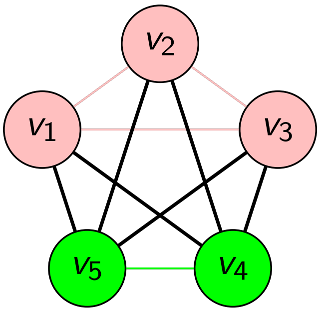


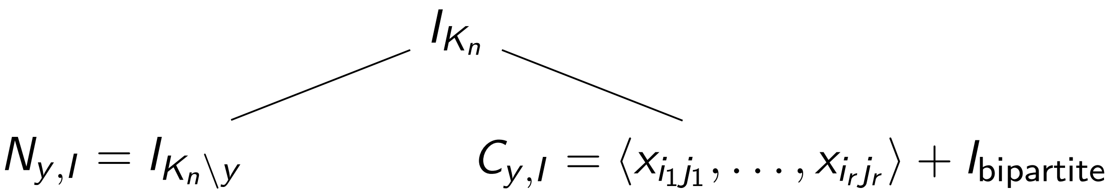
$$(g, e, f, a) \quad \rightarrow \quad fg - ae \in I_G$$

$$(g, e, f, a, g, d, c, b) \quad \rightarrow \quad cfg^2 - abde \in I_G$$

- Primitive closed even walks define a universal Gröbner basis for I_G.
- If I_G possesses a squarefree degeneration, then R/I_G is both normal and Cohen-Macaulay.
- $N_{y,I_G} = I_{G\setminus y}$

I_{K_5}	$C_{x_{45},I}$
$x_{12}x_{34} - x_{14}x_{23}$	~~$x_{12}x_{34} - x_{14}x_{23}$~~
$x_{13}x_{24} - x_{14}x_{23}$	~~$x_{13}x_{24} - x_{14}x_{23}$~~
$x_{12}x_{35} - x_{15}x_{23}$	~~$x_{12}x_{35} - x_{15}x_{23}$~~
$x_{13}x_{25} - x_{15}x_{23}$	~~$x_{13}x_{25} - x_{15}x_{23}$~~
$x_{12}x_{45} - x_{15}x_{24}$	x_{12}
$x_{14}x_{25} - x_{15}x_{24}$	$x_{14}x_{25} - x_{15}x_{24}$
$x_{13}x_{45} - x_{15}x_{34}$	x_{13}
$x_{14}x_{35} - x_{15}x_{34}$	$x_{14}x_{35} - x_{15}x_{34}$
$x_{23}x_{45} - x_{25}x_{34}$	x_{23}
$x_{24}x_{35} - x_{25}x_{34}$	$x_{24}x_{35} - x_{25}x_{34}$





I_{K_n}

$N_{y,I} = I_{K_n \setminus y}$ $\qquad$ $C_{y,I} = \langle x_{i_1 j_1}, \ldots, x_{i_r j_r} \rangle + I_{\text{bipartite}}$

- $I + J \in \mathbb{K}[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is GVD iff I is GVD in $\mathbb{K}[x_1, \ldots, x_r]$ and J is GVD in $\mathbb{K}[y_1, \ldots, y_s]$.
- $\langle x_{i_1 j_1}, \ldots, x_{i_r j_r} \rangle$ is GVD by definition.
- $I_{\text{bipartite}}$ is GVD since all toric ideals of bipartite graphs are GVD.
- $I_{K_n \setminus y}$ is GVD by induction.

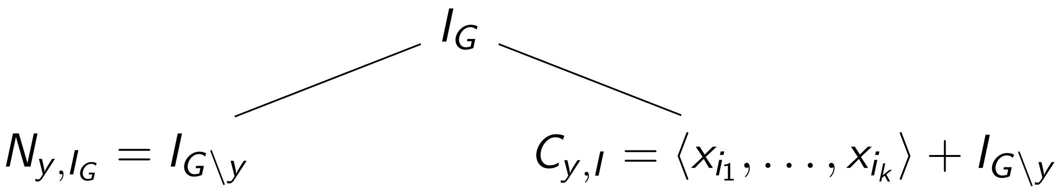
Theorem

The toric ideal of a complete graph I_{K_n} is GVD.

Squarefree Degenerations

Main Goal: Classify which toric ideals I_G are GVD.

If I_G has a quadratic Gröbner basis, then we get a similar setup:



I_G

$N_{y,I_G} = I_{G\setminus y}$ $\qquad$ $C_{y,I} = \langle x_{i_1}, \ldots, x_{i_k} \rangle + I_{G\setminus y}$

I_G has a quadratic GB $\Rightarrow$ $\mathbb{K}[G]$ is Koszul $\Rightarrow$ I_G is quadratically generated

Characterization of when I_G has a quadratic generating set was shown by Hibi, Nishiyama, Ohsugi, and Shikama.

Question: What about I_G which have a squarefree degeneration?

GVD Implications

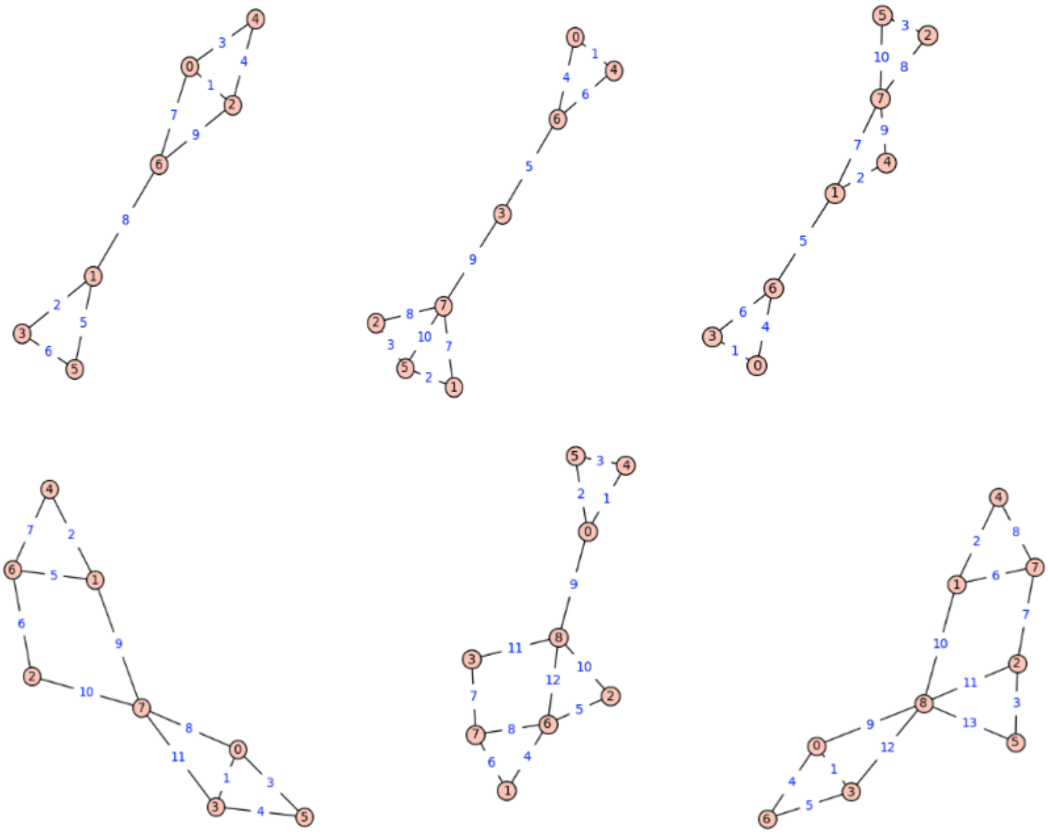
$$\begin{tikzcd}
\text{GVD} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{weakly GVD} \arrow[d, Rightarrow] \\
\text{GVD allowing sub} \arrow[r, Rightarrow, "\text{\& radical}"] & \text{weakly GVD allowing sub}
\end{tikzcd}$$

Each version of GVD $+$ homogeneous $\Longrightarrow$ glicci $\Longrightarrow$ Cohen-Macaulay

Can we classify all toric ideals of graphs which are:

- GVD up to substitution but not GVD
- Weakly GVD but not GVD allowing substitution
- Weakly GVD allowing substitution but not weakly GVD
- Weakly GVD allowing substitution but not GVD allowing substitution

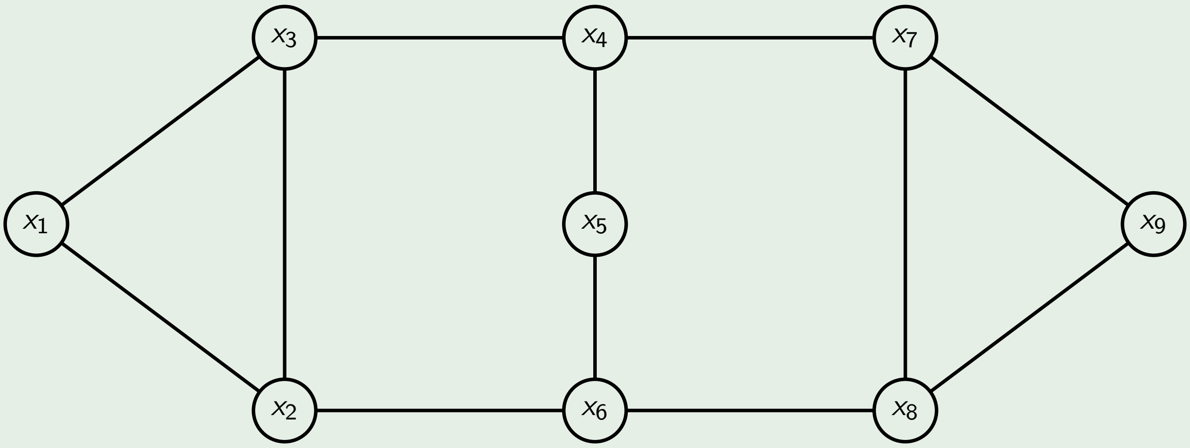
GVD up to Substitution but not GVD Examples



A Graph That Doesn't Fit

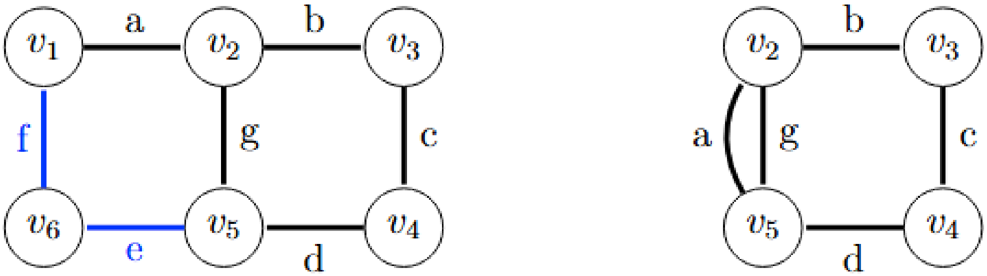
Example (Hà, Kara, O'Keefe)

The graph G is Cohen-Macaulay but $H = G \setminus x_5$ is not.



- I_G is not GVD in any sense.
- I_G does not have any lexicographic squarefree degeneration.
- I_G is glicci (which is good news for a conjecture from liaison theory).

Graph Operations: Star Contractions



$$\langle ace - bdf, ae - fg, bd - cg\rangle \quad \longrightarrow \quad \langle ac - bd, a - g, bd - cg\rangle$$

- If v has degree 2 before and after, then I_{G_v} is GVD iff I_G is GVD.
- Under certain squarefree assumptions, if I_G is GVD then so is I_{G_v}.

Open Questions

1. The toric ideal of a graph I_G is normal iff G satisfies the odd cycle condition (every disjoint pair of odd cycles in G should be connected by at least one edge). If I_G is normal, then it is also Cohen-Macaulay. Is it true that all such toric ideals are also GVD? Can you prove this for a special family of graphs?
2. If I_G is the toric ideal of a graph G, then for any choice y, $C_{y,I_G} = M + I_{G\setminus y}$ where M is a monomial ideal. Provide a graph-theoretic description of M.
3. Let I_G be the toric ideal of a graph. Prove that there always exists at least one edge $y \in E(G)$ such that $\mathrm{init}_y(I_G) = C_{y,I_G} \cap (N_{y,I_G} + (y))$.
4. (Open ended) Which graph operations on G preserve the GVD property for I_G? This general question is towards the direction of classifying which toric ideals are GVD.

Thank you!

7.3 Tutorial Problems

Here are the associated tutorial problems.

GVD SMS mini-course: Gröbner Geometry and Applications

Sergio Da Silva and Patricia Klein

June 2025

1 Exercises

Exercises marked with a $*$ require commutative algebra background beyond what has been covered in this summer school and what would ordinarily be covered in a first semester commutative algebra course. Please feel free to ask for hints (on anything, but especially on these)!

Throughout, assume that I is an unmixed, homogeneous ideal of the polynomial ring $R = \kappa[x_1, \ldots, x_n]$, that y is a variable of R, that $<$ is a lexicographic term order with y the largest variable, and that

$$\mathcal{G} = \{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$$

is a $<$-Gröbner basis of I. Assume that $\operatorname{init}_y(I) = C_{y,I} \cap (N_{y,I} + (y))$ is a geometric vertex decomposition. Assume that the geometric vertex decomposition is nondegenerate unless otherwise specified.

1. Show that the "bow-tie" simplicial complex Γ whose Stanley-Reisner ideal is $I_\Gamma = (ab, ae, bd, de) \subset \kappa[a, b, c, d, e]$ is not vertex decomposable.
2. (a) Show that $\{h_1, \ldots, h_\ell\}$ is a $<$-Gröbner basis of $N_{y,I}$, that $\{q_1, \ldots, q_k, h_1, \ldots, h_\ell\}$ is a $<$ Gröbner basis of $C_{y,I}$, and that $\{yq_1, \ldots, yq_k, h_1, \ldots, h_\ell\}$ is a Gröbner basis of $\operatorname{init}_y(I)$.
   (b) Show that $C_{y,I} = (\operatorname{init}_y(I) : y)$ and that $N_{y,I} + (y) = \operatorname{init}_y(I) + (y)$.
   (c) Show that $\operatorname{init}_<(\operatorname{init}_y(I)) = \operatorname{init}_<(I)$.
   (d) Show that the Hilbert function of I agrees with the Hilbert function of $\operatorname{init}_y(I)$.
3. Suppose that J is the Stanley–Reisner ideal of Γ, where the variable x_i of R corresponds to the vertex i. Set $R' = \kappa[x_1, \ldots, x_{n-1}]$. Interpreting $\operatorname{del}_\Gamma(n)$ and $\operatorname{lk}_{\Gamma(n)}$ as complexes on $\{1, \ldots, n-1\}$, show that $I_{\operatorname{del}_\Gamma(i)} = N_{x_i,J}$ and that $I_{\operatorname{lk}_\Gamma(i)} = C_{x_i,J}$.
4. Show the following:
   (a) For $f \in R$ and $t \geq 1$, $\operatorname{init}_y(f^t) = (\operatorname{init}_y(f))^t$.
   (b) If I is radical (resp. prime), then $N_{y,I}$ is radical (resp. prime).
   (c) If $\operatorname{init}_y(I)$ is radical, then I is radical. (Optional: This is an example of a much more general phenomenon. What are other things that you can replace $\operatorname{init}_y(I)$ by?) Give an example to show that the converse is not true.
   (d) If I is geometrically vertex decomposable, then I is radical.
5. The purpose of this problem is to show that J admits a geometric vertex decomposition at y if and only if J has a generating set that is linear in y. Let J be an ideal of R. Show the following:
   (a) If J has a generating set in which each generator is at most linear in y, then the $<$-Gröbner basis of J also satisfies this property.

   (b) Let $C = \bigcup_t (\mathrm{init}_y(C) : y^t)$ and $N = (f \in J \mid y \text{ does not divide any term of } f)$. Show that $\mathrm{init}_y(J) = C \cap (N + (y))$ if and only if J has a generating set in which each generator is at most linear in y.

6. Prove that an ideal $J \subset \kappa[x]$ is geometrically vertex decomposable if and only if $J = (ax + b)$ for some $a, b \in \kappa$.

7. Consider the ideal $J = (aeg - bcf, ae - bd, cf - dg) \subset R = \kappa[a, \ldots, g]$ which is generated by a universal Gröbner basis. Prove that R/J is Cohen-Macaulay using geometric vertex decomposability. (You may also choose to carefully choose some monomial order $<$ on R so that $\mathrm{init}_<(J)$ has a vertex decomposable Stanley-Reisner complex.)

8. Let X be a generic $m \times n$ matrix (i.e., a matrix whose ij^{th} entry is the variable z_{ij}). Assume $m \leq n$. Let $<$ be the lexicographic order on $z_{mn} > z_{mn-1} > \cdots > z_{m1} > z_{m-1n} > \cdots > z_{11}$. Let J be the ideal generated by the size m-minors of X. You may assume that the size m-minors of X form a $<$-Gröbner basis. Show that J is geometrically vertex decomposable.

9. (Harder) Let X be a generic $m \times n$ matrix, and let J be the ideal generated by the size k-minors for some $k \leq m \leq n$. Show that J is geometrically vertex decomposable. (Hint: The statement you've been asked to show is not strong enough for an induction to go through. You'll need to expand the class of ideals. Part of the exercise is formulating the correct statement. We encourage you to check with instructors as you consider statements for the induction. Perhaps start with the ideal of size 2 minors of a generic 3×3 matrix.)

10. * Let J be a homogeneous ideal of R. Prove that, if R/J is normal, then J is prime.

11. * Prove that, if I is geometrically vertex decomposable and R/I is regular in codimension 1, then I is prime.

12. * Let X be a generic $m \times n$ matrix, and let J be the ideal generated by the size k-minors for some $k \leq m \leq n$. Show that R/J is regular in codimension 1. Conclude, using previous exercises, that J is prime.

13. * Describe the Hilbert series of R/I in terms of the Hilbert series of $R/C_{y,I}$ and $R/N_{y,I}$. How does the degree of $\mathbb{V}(I)$ compare to the degrees of $\mathbb{V}(C_{y,I})$ and $\mathbb{V}(N_{y,I})$?

14. If I_Γ is the Stanley–Reisner ideal of the simplicial complex Γ, then I_Γ is geometrically vertex decomposable if and only if Γ is vertex decomposable.

15. Suppose that I is radical and that the geometric vertex decomposition of I at y is degenerate. If $C_{y,I} = (1)$, then I contains an element of the form $y - f$ for some $f \in R$ with no term divisible by y. If $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$, then I has a generating set in which no term of any generator is divisible by y.

16. Let $J = (y(zs - x^2), ywr, wr(z^2 + zx + wr + s^2))$.

    (a) Show that J is geometrically vertex decomposable.

    (b) Show that J has no squarefree initial ideals. (Hint: From Part (a) you know that J is radical. Therefore, you cannot prove Part (b) by showing that J is not radical.)

17. Show that $J = (xy - z^2)$ is geometrically vertex decomposable, but $J = (x^2 - z^2)$ is not (both as ideals in $\kappa[x, y, z]$). Can you guess what condition guarantees that a principal ideal is geometrically vertex decomposable?

18. Show that even though the ideal $I = (x^2 - y^2) \in \mathbb{C}[x, y]$ is not geometrically vertex decomposable, there is a linear change of coordinates φ defined by

$$r = c_1 x + c_2 y$$

$$s = c_3 x + c_4 y$$

where $\varphi(I) \subset \mathbb{C}[r, s]$ is geometrically vertex decomposable. Here $c_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$ and the 2×2 coefficient matrix is invertible. Can you completely characterize when $\varphi(I)$ is geometrically vertex decomposable? (This question shows that being geometrically vertex decomposable is not an invariant and highly depends on the coordinates.)

19. Show that the maximal minors of a generic matrix X form a Gröbner basis under any diagonal term order, i.e., a term order in which the initial term each minor is the product of entries along the antidiagonal of the corresponding submatrix X.

20. (Harder) Let X be a generic $m \times n$ matrix. Fix $k \leq m \leq n$. Show that the size k minors of X form a Gröbner basis of the ideal they generate under any diagonal term order.

21. Given the toric ideal I_G, prove that $N_{y,I} = I_{G \setminus y}$ where $G \setminus y$ is formed from G by deleting the edge y.

22. Pick your favourite graph G and check whether it is geometrically vertex decomposable (hopefully your favourite graph has more than two cycles).

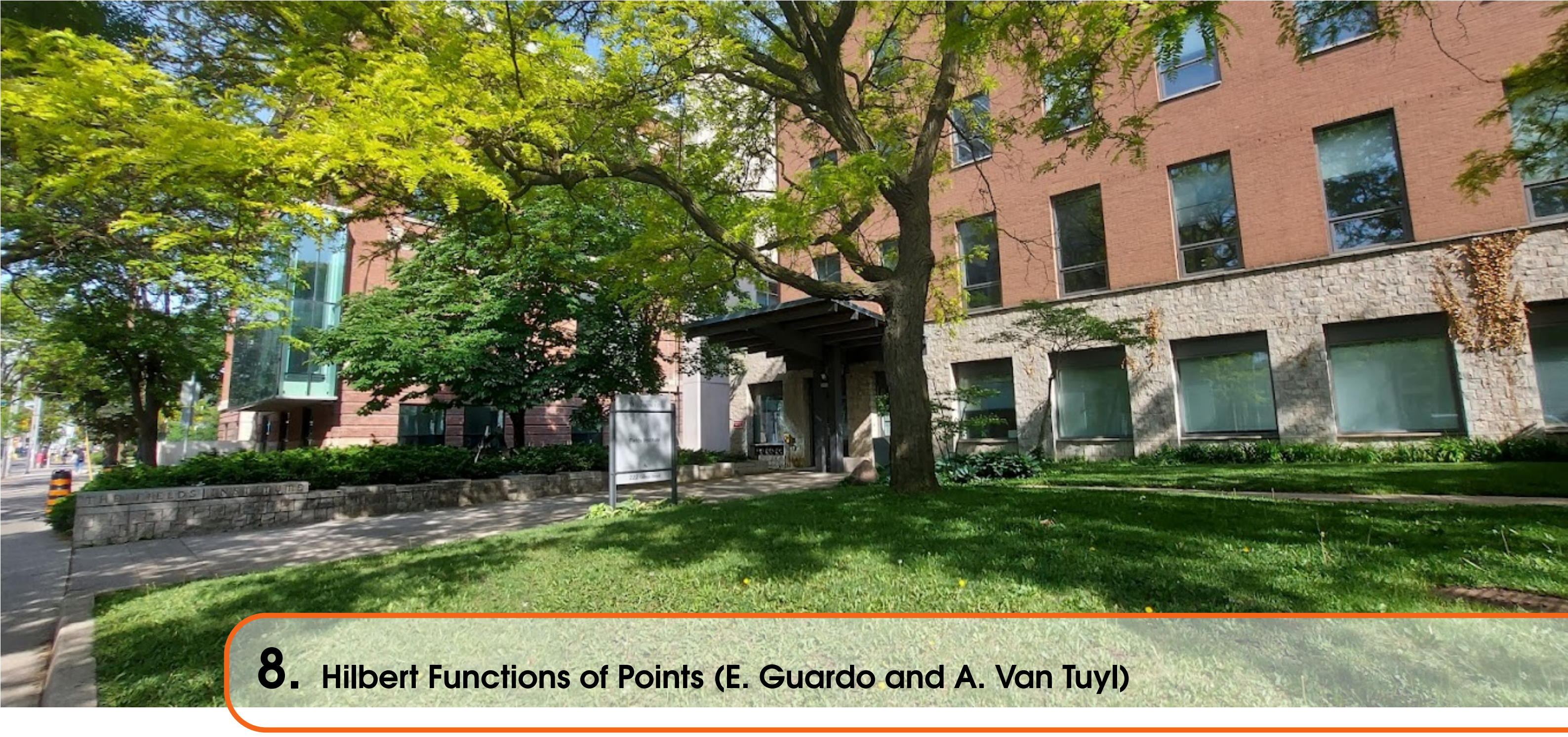
23. Some open questions about toric ideals of graphs:

    (a) The toric ideal of a graph G is normal if and only if G satisfies the odd cycle condition (every disjoint pair of odd cycles in G should be connected by at least one edge). If I_G is normal, then it is also Cohen-Macaulay. Is it true that all such toric ideals are also geometrically vertex decomposable? Can you prove this for a special family of graphs?

    (b) If I_G is the toric ideal of a graph G, then for any choice y, $C_{y,I_G} = M + I_{G \setminus y}$ where M is a monomial ideal. Provide a graph theoretic description of M.

    (c) Let I_G be the toric ideal of a graph. Prove that there always exists at least one edge $y \in E(G)$ such that $\text{init}_y(I_G) = C_{y,I_G} \cap (N_{y,I_G} + (y))$.

    (d) (Open ended) Which graph operations on G preserve the geometrically vertex decomposable property for I_G? This general question is towards the direction of classifying which toric ideals are geometrically vertex decomposable.



8. Hilbert Functions of Points (E. Guardo and A. Van Tuyl)

This course focuses on the study of homological questions and invariants of points in projective space (e.g., Hilbert functions, resolutions, regularity). Students saw the interplay between classical algebraic geometry and commutative algebra, e.g., how to use the Hilbert function to deduce geometric information about the set of points. This course was taught by Adam Van Tuyl (McMaster) and Elena Guardo (Catania).

8.1 Video Links

You can watch the original lectures using the following links:

- Lecture 1 Video
- Lecture 2 Video
- Lecture 3 Video

8.2 Lecture Notes

We have included copies of Elena and Adam's lecture notes and her tutorials. Elena and Adam have included scanned copies of their handwritten lecture notes.

Lecture 1 Hilbert functions of reduced sets of points (Fields-SMS Summer School, June 2025)

References: • Bruns-Herzog (Chap 4)
· Geramita, Maroscia, Roberts, J. Lond Math Soc, 1983

0. Points in $\mathbb{P}^n$

Fix $k=\mathbb{C}$ and $R=k[x_0,..,x_n]$

$\mathbb{P}^n = k^{n+1}\setminus\{(0,..,0)\}/\sim$ ← $\sim$ is an equivalence relation,

where $(a_0,..,a_n)\sim(b_0,..,b_n) \iff$ there exists $\lambda\neq 0$ such that
$(b_0,..,b_n)=(\lambda a_0,..\lambda a_n)$

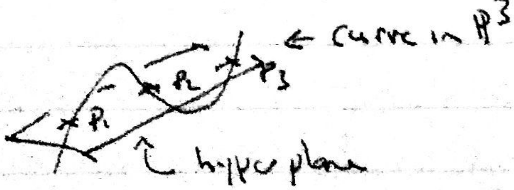
HW Verify $\sim$ is an equivalence relation

$[a_0:\cdots:a_n]$ denotes the equivalence class of $(a_0,..,a_n)$. This is a point in $\mathbb{P}^n$

Object of study: finite sets of points $\mathbb{X}=\{P_1,..,P_s\}\subseteq\mathbb{P}^n$

Motivation · "simplest" geometric object
· arise when we consider "hyperplane sections"

e.g



points of intersection give info about original variety

· "interpolation problem" find curves, surfaces, etc that pass through given $\mathbb{X}\subseteq\mathbb{P}^n$

I. Ideal of points

· $f\in R$ homogeneous if all terms have same degree

Ex $R = k[x_0, x_1, x_2]$ $\quad f(x_0,x_1,x_2) = 3x_0^3 + 4x_0x_1x_2 + 17x_1x_2^2 \leftarrow$ homog

$g(x_0,x_1,x_2) = 12x_0^3 + 2025x_2^2 + 7 \leftarrow$ not homog

Def^n Ideal of point $P \in \mathbb{P}^n$

$I(P) = \langle f \mid f \in R$ homog and $f(P) = 0 \rangle$

$\boxed{\text{HW-Fact}}$ $I(P)$ homog ideal (generated by homog elements)

Note f homog is required to make $f(P)=0$ well defined since P is equivalence class, so need $f(P)=0$ for all elements in class

Ex $P = [1:2:3]$ and $f(x_0,x_1,x_2) = 6x_0^2 - 2x_0x_2 - 3x_0x_1 + x_1x_2 \leftarrow$ homog of deg

$f(1,2,3) = 6\cdot 1^2 - 2\cdot 1\cdot 3 - 3\cdot 1\cdot 2 + 2\cdot 3 = 0$

and $f(\lambda\cdot 1, \lambda 2, \lambda\cdot 3) = 6\cdot(\lambda\cdot 1)^2 - 2(\lambda\cdot 1)(\lambda\cdot 3) - 3(\lambda\cdot 1)(\lambda\cdot 2) + (\lambda 2)(\lambda 3)$

$= \lambda^2 \cdot 0 = 0$

So $f \in I(P)$.

$\boxed{\text{HW-Fact}}$ If $P = [a_0 : \cdots : a_n]$ with $a_0 \neq 0$, then

$I(P) = \langle a_0x_1 - a_1x_0,\ a_0x_2 - a_2x_0,\ \ ,\ a_0x_n - a_nx_0 \rangle$

Ex $I([1:2:3]) = \langle x_1 - 2x_0,\ x_2 - 3x_0 \rangle$

Recall
- $\bar{0} \neq \bar{L} \in R/J$ is a nonzerodivisor of R/J if whenever $LF \in J$, $F \in J$
- $\bar{L}_1, \ldots, \bar{L}_s$ is a regular sequence if $\bar{L}_i$ is a nonzerodivisor of $R/(J, L_1, \ldots, L_{i-1})$
- J is a complete intersection if $J = \langle L_1, \ldots, L_s \rangle$ and $L_1, \ldots, L_s$ is a regular sequence on $R/\langle 0 \rangle \cong R$
- $\operatorname{depth}(R/J)$ = length of longest regular seq in R/J
- $\dim(R/J)$ = length of longest chain of prime ideals in R/J

Hw-Facts · $I(P)$ is prime ideal
· $I(P)$ is a complete intersection
· $\dim R/I(P) = 1 = \operatorname{depth}(R/I(P)) \Rightarrow \boxed{R/I(P) \text{ is Cohen Macaulay}}$

Def^n Given $\mathbb{X} = \{P_1, \dots, P_s\}$, ideal of $\mathbb{X}$ $\quad I_{\mathbb{X}} = I(P_1) \cap \dots \cap I(P_s)$

Rmk $F \in I_{\mathbb{X}} \Leftrightarrow F(P_i) = 0$ for all $i = 1, \dots, s$

Lemma Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$. Then exists nonzero divisor $\bar{L} \in R/I_{\mathbb{X}}$ with $\deg \bar{L} = 1$

Proof For any $\mathbb{X} \subseteq \mathbb{P}^n$, can find hyperplane $H \subseteq \mathbb{P}^n$ such that $H \cap \mathbb{X} = \emptyset$ (e.g. in $\mathbb{P}^2$, find a line that misses the points). So there exists $L \in R$, L homog and $\deg L = 1$ such that
$H = V(L) = \{Q \in \mathbb{P}^n \mid L(Q) = 0\}$ ← vanishing ideal of H
Since $H \cap \mathbb{X} = \emptyset$, $L(P) \neq 0$ for all $P \in \mathbb{X}$. So $\bar{L} \neq \bar{0}$ in $R/I_{\mathbb{X}}$.
If $LF \in I_{\mathbb{X}}$, then $(LF)(P) = L(P)F(P) = 0$ for all $P \in \mathbb{X}$.
But $L(P) \neq 0$, so $F(P) = 0$. So $F \in I_{\mathbb{X}}$, i.e. $\bar{L}$ is a nonzero divisor ☒

Hw-Facts · $I_{\mathbb{X}}$ radical $\Leftrightarrow \sqrt{I_{\mathbb{X}}} = I_{\mathbb{X}}$
· $\dim R/I_{\mathbb{X}} = 1 = \operatorname{depth} R/I_{\mathbb{X}}$

Remark $I_{\mathbb{X}}$ has properties similar to square-free mon ideal
$\Rightarrow$ both radical & intersection of prime ideals that are also complete intersect

II Hilbert fcns & Macaulay's Thm

Fix $d \in \mathbb{N}$ $\quad R_d = \{f \in R \mid f \text{ homog of deg } d\}$
If I homog, $I_d = I \cap R_d = \{f \in I \mid f \text{ homog of deg } d\}$

[HW-Fact] R_d is a vector space over k, and $I_d \subseteq R_d$ is a subspace

Note To make this work, convention $0 \in R_d$ for all d

Fact If $R = k[\underbrace{x_0,\dots,x_n}_{n+1}]$, then $\dim_k R_d = \binom{n+d}{d}$

Why? Basis for R_d is all monomials of degree d

$\#\{x_0^{d_0}\cdots x_n^{d_n} \mid d_0+\dots+d_n = d\} = \#$ sol^{n}s to $z_0+\dots+z_n = d$ $z_i \geq 0$

$\Leftrightarrow$ # ways to arrange n "bars" and d "stars" $\Leftrightarrow \binom{n+d}{d}$

Ex $R = k[x_0,x_1,x_2]$ $d=5$

$x_0^2x_1^2x_2 \leftrightarrow 2+2+1=5 \leftrightarrow **|**|*$

$x_0^2x_2^3 \leftrightarrow 2+0+3=5 \leftrightarrow **||***$

_ _ _ _ _ _ _

↑ placing 5 stars and 2 bars into 5+2 spots $= \binom{5+2}{5} = 21$

Def^n Let $I \subseteq R$ be a homog ideal. Hilbert function of R/I

$$H_{R/I}: \mathbb{N} \to \mathbb{N}$$
$$d \mapsto \dim_k R_d - \dim_k I_d = \binom{d+n}{n} - \dim_k I_d$$

Convention Write $h_0 h_1 h_2 \dots$ or $\{h_i\}_{i \geq 0}$

Def^n (i^{th} binomial expansion of a) Given integers on a and i, a can be written uniquely as $a = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j}$

with $m_i > m_{i-1} > \dots > m_j \geq j$

Define a function $\_^{\langle i\rangle} : \mathbb{N} \to \mathbb{N}$

$$a = \binom{m_i}{i} + \cdots + \binom{m_j}{j} \mapsto a^{\langle i\rangle} = \binom{m_i+1}{i+1} + \cdots + \binom{m_j+1}{j+1}$$

Ex $a=17, i=3$

$$a = 17 = \binom{5}{3} + \binom{4}{2} + \binom{1}{1} = 10+6+1$$

$$a^{\langle 3\rangle} = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} = 15+10+1 = 26$$

Def^n A seq of nonnegative integers $\{c_i\}_{i\geq 0}$ is an O-sequence if $c_0 = 1$ and $c_{i+1} \leq c_i^{\langle i\rangle}$ for all $i \geq 1$

[MACAULAY'S THM] TFAE

- $\{h_i\}_{i\geq 0}$ is the Hilbert function of some ring of the form $\frac{k[x_0,\ldots,x_n]}{I}$
- $\{h_i\}_{i\geq 0}$ is an O-sequence

Ex 1 4 10 17 27 h_5 h_6 ... cannot be $\overset{a}{\text{the}}$ HF since $h_3 = 17$, so $h_4 \leq h_3^{\langle 3\rangle} = 17^{\langle 3\rangle} = 26$. But $h_4 = 27$. Not an O-se

III Hilbert fcns of points

Def^n Given $H = \{h_i\}_{i\geq 0}$, first difference fcn is $\Delta H(i) = h_i - h_{i-1}$ with $h_{-1} = 0$

Ex $H = 1\ 4\ 10\ 19\ 30\ 40\ 50 \cdots$ ← increase by 10
$\Delta H = 1\ 3\ 6\ 9\ 11\ 10\ 10 \cdots$

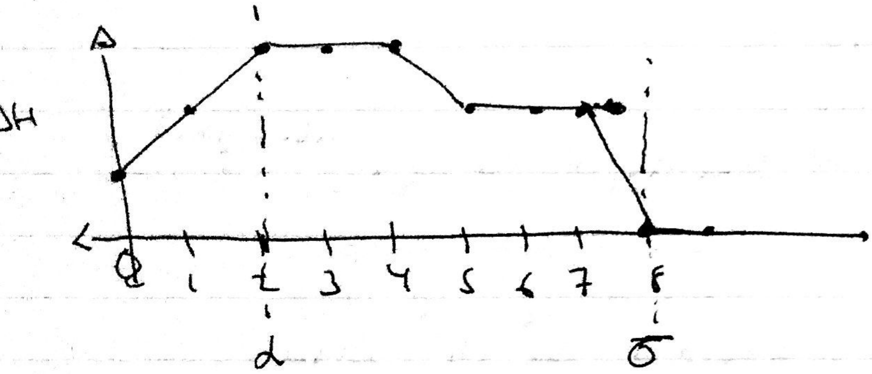
Thm (Geramita-Maroscia-Roberts '83)
$H = \{h_i\}_{i\geq 0}$ is the Hilbert function of s points in $\mathbb{P}^n$ $\Longleftrightarrow$

1. $H(1) \leq n+1$
2. ΔH is an O-seq
3. $H(i) = s$ for <u>all</u> $i >> 0$

(Special case: Points in $\mathbb{P}^2$) ~~H is the HF~~ $H = \{h_i\}$ is the HF of s points in $\mathbb{P}^2$ $\iff$ there exists $\alpha < \sigma$ such that

- $\Delta H(i) = i+1$ for $0 \leq i < \alpha$
- $\Delta H(i+1) \leq \Delta H(i)$ for $\alpha \leq i < \sigma$
- $\Delta H(i) = 0$ for $\sigma \leq i$

Ex $H = 1\ 3\ 6\ 9\ 12\ 14\ 16\ 18\ 18 \to$
$\Delta H = 1\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 0 \to$



a valid HF!

$\alpha = 2$, $\sigma = 8$

Parts of the proof sketched
($H \Rightarrow \Delta H$ is a O-seq)
Let $L \in R/I_X$ be the nonzero divisor of the lemma. Have a s.e.s

$$0 \to R/I_X(-1) \xrightarrow{\times \bar{L}} R/I_X \to R/I_{X,L} \to 0$$

with degree 0 maps ($R/I_X(-1)$ same as ring R/I_X, but with grading shifted by 1). Hilbert fcns are additive on s.e.s so

$$H_{R/I_X}(i-1) - H_{R/I_X}(i) + H_{R/I_{X,L}}(i) = 0 \quad \forall i$$

$\iff H_{R/(I_X,L)}(i) = H_{R/I_X}(i) - H_{R/I_X}(i-1) = \Delta H_{R/I_X}(i)$

So $\Delta H_{R/I_X}$ is the HF of $R/I_X,L$, and thus by Macaulay's thm, $\Delta H_{R/I_X}$ must be an O-sequence

($H \Rightarrow H(1) \leq n+1$)

$I_X = I(P_1) \cap \dots \cap I(P_s) \subseteq R = k[x_0, \dots, x_n]$

So $H_{R/I_X}(1) = \dim_k R_1 - \dim_k (I_X)_1 \leq \dim_k R_1 = n+1$

($H \Rightarrow H$ eventually constant) Since L is a nonzero divisor and $\dim R/I_X = 1$, $\dim R/(I_X,L) = 0$, i.e. $R/I_X,L$ is an artinian ring.

So, there exists t such that

$(I_X, L)_{t+i} = R_{t+i}$ for all $i \geq 0$

So $H_{R/(I_X,L)}(t+i) = 0$ for all $i \geq 0$

This implies $H_{R/I_X}(t+1) = H_{R/I_X}(t+2) = \dots$

(Note that this only shows H_{R/I_X} must be eventually constant. Need more work to show this constant is $|X| = s$)

FAT POINTS in $\mathbb{P}^2$

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IN THIS LECTURE WE GIVE A SUMMARY OF WORKS DONE BY MANY PEOPLE ON THE SUBJECT "FAT POINTS", i.e., CERTAIN 0-DIMENSIONAL PROJECTIVE SCHEMES GIVEN BY THE INTERSECTIONS OF POWERS OF IDEALS OF POINTS. WE RESTRICT OUR ATTENTION TO $\mathbb{P}^2$ SINCE THERE ARE A LOT OF UNANSWERED QUESTIONS.

WE WORK IN $\mathbb{P}^2_k$ WITH k AN ALGEBRAICALLY CLOSED FIELD OF CHARACTERISTIC ZERO. SET $R = k[x,y,z]$ THE POLYNOMIAL RING IN x,y,z.

DEFINITION: GIVEN DISTINCT POINTS $X = \{P_1,\dots,P_s\}$ in $\mathbb{P}^2 := \mathbb{P}^2_k$ AND POSITIVE INTEGERS $m_1,\dots,m_s$ THE SCHEME DEFINED BY THE IDEAL

$$I_Z = \mathfrak{p}_1^{m_1} \cap \mathfrak{p}_2^{m_2} \cap \dots \cap \mathfrak{p}_s^{m_s};$$

WHERE $\mathfrak{p}_i$ is THE PRIME IDEAL CORRESPONDING TO P_i, IS CALLED A FAT POINT SCHEME (OR SET OF FAT POINTS) WITH MULTIPLICITY SET $\{m_1,\dots,m_s\}$ SUPPORTED on X

WE ALSO DENOTE IT BY $Z = m_1P_1 + \dots + m_sP_s$
THE SUPPORT OF Z CONSISTS OF THE POINTS P_i FOR WHICH m_i IS POSITIVE. EACH POINT P_i OF MULT. m_i HAS DEGREE $\binom{m_i+1}{2}$

THE HILBERT POLYNOMIAL OF Z IS (RECALL F. GALETTO'S LECTURE)

$$p(Z) = \sum_{i=1}^{s} \binom{m_i+1}{2} = \deg Z$$

WE HAVE $\dim_k (I_Z)_i = \binom{i+2}{2} - \sum_i \binom{m_i+1}{2}$
and the HILBERT FUNCTION OF Z IS

$$(H_{R/I_Z})_i = \dim_k R_i - \dim_k (I_Z)_i = \binom{i+2}{2} - \dim_k (I_Z)_i$$

FAT POINTS SCHEMES HAVE PLAYED AN IMPORTANT ROLE IN NUMEROUS PROBLEMS FROM MANY BRANCHES OF MATHEMATICS.

WHEN EACH $m_i = 1$, THAT IS, WHEN WE HAVE SIMPLE POINS, MACAULAY'S THEOREM AND PAPERS FROM GERAMITA-MAROSCIA-ROBERTS / GIUFFRIDA-MAGGIONI-RAGUSA COMPLETELY CHARACTERIZE THE HILBERT FUNCTIONS

IF $m_i > 1$ FOR SOME i
$\Rightarrow$ THERE ARE STILL A LOT OF OPEN PROBLEMS

GERAMITA-MIGLIORE-SABOURIN RAISE THE QUESTION OF FINDING ALL HILBERT FUNCTIONS AND GRADED BETTI NUMBERS FOR IDEALS OF DOUBLE POINTS SUBSCHEMES $2p_1 + \dots + 2p_s \subset \mathbb{P}^2$ FOR ALL CONFIGURATIONS

$2p_1 + \dots + 2p_s \subset \mathbb{P}^2$ FOR ALL CONFIGURATIONS OF THE POINTS P_i

ONE GOAL IS TO RESTRICT THE PROBLEM IN CASE THAT THE GEOMETRY OF THE SUPPORT IS KNOWN OR FOR A SMALL NUMBER OF POINTS IN "GENERIC" POSITION.

WHAT MAKES INTERESTING THESE SCHEMES IS THE FACT THAT EACH VECTOR SPACE I_t GIVES THE LINEAR SYSTEM on $\mathbb{P}^2$ CONSISTING OF ALL CURVES OF DEGREE t HAVING AT LEAST MULTIPLICITY m_i AT EACH P_i.

A POINT OF MULTIPLICITY m IMPOSES $\binom{m+1}{2} = \frac{m(m+1)}{2}$ CONDITIONS on CURVES. So, IF THE POINTS IMPOSE INDIPENDENT CONDITIONS, THE DIMENSION OF OUR LINEAR SYSTEM IN DEGREE t

$$(\dim_k R_t - p(z))^+ = \left(\binom{t+2}{2} - \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2} \right)^+ =$$

$=$ VIRTUAL DIMENSION of THE SYSTEM I_t $(\mathrm{v.}\dim I_t)$

WHERE $x^+ = \begin{cases} x & \text{IF } x \geqslant 0 \\ 0 & \text{OTHERWISE} \end{cases}$.

WHEN $\dim_k I_t = \mathrm{vdim}\, I_t$

THE SYSTEM IS CALLED REGULAR OR NON SPECIAL

IF $\dim_k I_t = \mathrm{vdim}\, I_t + S$ with $S \geqslant 0$

THE NUMBER S is CALLED THE SUPERABUNDANCE

THE NUMBER S IS CALLED THE SUPERABUNDANCE OF I_t

EXAMPLE 1

CONSIDER THE FOLLOWING EXAMPLE:
LET $P_1, P_2 \in \mathbb{P}^2$ TWO POINTS AND $X = \{P_1, P_2\}$
BE THE SUPPORT OF THE FAT POINT SCHEME

$$Z = 2P_1 + 2P_2$$

WE HAVE $\deg Z = 2\binom{2+1}{2} = 2 \cdot \binom{3}{2} = 6$

WE WANT TO COMPUTE $(H_{R/I_Z})_t$

t	0	1	2	3	4	5	$\longmapsto$
$(H_{R/I_Z})_t$	1	3	5	6	6	$\mapsto$	
$\Delta H_{R/I_Z}$	1	2	2	1	0	$\mapsto$	

$\Longleftrightarrow \dim_K (I_Z)_2 = 6-5 = 1$

BUT IF WE COMPUTE THE $\operatorname{Vdim} I_Z$ WE HAVE

$$\operatorname{Vdim} (I_Z)_2 = \binom{2+2}{2} - \sum_{i=1}^{2} \frac{m_i(m_i+1)}{2} =$$
$$= \binom{4}{2} - 2 \cdot 3 = 6 - 6 = 0$$

BUT THIS IS NO TRUE SINCE THERE IS THE CONIC C DEFINED BY THE L^2, WHERE L IS THE LINE THAT PASSES THROUGH P_1 AND P_2.

IN A FEW WORDS WE HAVE SOME UNEXPECTED CURVES!
AND THIS EXAMPLE IS ONE OF THE EXCEPTIONAL

IN A FEW WORDS WE HAVE SOME UNEXPECTED CURVES.
AND THIS EXAMPLE IS ONE OF THE EXCEPTIONAL CASES STUDIED BY ALEXANDER-HIRSCHOWITZ
SPECIFICALLY, THE A-H THEOREM SAYS THAT A GENERAL COLLECTION OF S DOUBLE POINTS in $\mathbb{P}^m$ IMPOSES INDIPENDENT CONDITIONS on HOMOGENEOUS POLYNOMIALS OF DEGREE d WITH A LIST OF EXCEPTIONS.

THEOREM [A-H]. LET X be A GENERAL COLLECTION OF S DOUBLE POINTS in $\mathbb{P}^m_k$ OVER AN ALG. CLOSED FIELD OF $\mathrm{char}(k)=0$. LET $R_d := k[x_1, \dots, x_m]_d$ THE SPACE OF HOMOGENEOUS POLYNOMIALS OF DEGREE d. LET $I_X(d) \subseteq R_d$ BE THE SUBSPACE OF POLYNOMIALS THROUGH X, WITH ALL FIRST PARTIAL DERIVATIVES VANISHING AT THE POINTS OF X. THEN THE SUBSPACE $I_X(d)$ HAS THE EXPECTED CODIMENSION

$$\min\left((m+1)s, \binom{m+d}{m}\right)$$

EXCEPT IN THE FOLLOWING CASES

(1) $d=2 \quad 2 \leq s \leq m$

(2) $m=2$, $d=4$, $s=5$; → EXAMPLE 1

(3) $m=3$, $d=4$, $s=9$;

(4) $m=4$, $d=3$, $s=7$

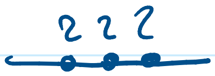
(5) $m=4$, $d=4$, $s=14$

THIS THEOREM HAS AN EQUIVALENT FORMULATION IN TERMS OF HIGHER SECANT VARIETIES.
WE WILL REFER TO [FRANCESCO RUSSO]

① FOR MORE ON [A-H] THM SEE BRAMBILLA - OTTAVIANI

=

② EX $X_1 = \{P_1, P_2, P_3\}$



1 1 1 1 1 1
1 1 1

1 2 2 2 1 1

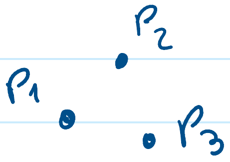
→ SOME OF THE MULT. 2+2+2

$I_{Z_1} = 2P_1 + 2P_2 + 2P_3$

SUPPORT IS A LINE L

t	0	1	2	3	4	5	6	7
$H_{R/I_{Z_1}}$	1	3	5	7	8	9	9	$\to$
$\Delta H_{R/I_{Z_1}}$	1	2	2	2	1	1	0	$\to$

③ EX $X_2 = \{P_1, P_2, P_3\}$



$I_{Z_2} = 2P_1 + 2P_2 + 2P_3$ SUPPORTED ON 3 GENERAL POINTS

by [AH] WE HAVE $\deg Z_2 = 9$

	0	1	2	3	4	
$H_{R/I_{Z_2}}$	1	3	6	9	9	$\to$

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S. COOPER - B. HARBOURNE - REGINA LECTURE ON FAT POINTS

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POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

mercoledì 21 maggio 2025 11:47

MAIN REFERENCE

E. GUARDO - A. VAN TUYL
ARITHMETICALLY COHEN-MACAULAY SETS OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$
SPRINGER 2015

BI PROJECTIVE SPACE

LET K BE AN ALGEBRAICALLY CLOSED FIELD OF $CHAR(k) = 0$

$$\mathbb{P}^1 \times \mathbb{P}^1 = \left\{ ((a_0,a_1),(b_0,b_1)) \in K^2 \times K^2 \;\middle|\; \begin{array}{l} \text{NEITHER } (a_0,a_1) = (0,0) \\ \text{NOR } (b_0,b_1) = (0,0) \end{array} \right\} \Big/ \sim$$

WHERE $\sim$ IS THE EQUIVALENCE RELATION
$((a_0,a_1),(b_0,b_1)) \sim ((c_0,c_1),(d_0,d_1))$
IF THERE EXIST NON-ZERO $\lambda, \mu \in k$ SUCH THAT
$(a_0,a_1) = (\lambda c_0, \lambda c_1)$ AND $(b_0,b_1) = (\lambda d_0, \lambda d_1)$
AN ELEMENT OF $\mathbb{P}^1 \times \mathbb{P}^1$ IS CALLED A POINT
AND WE DENOTE THE EQUIVALENCE CLASS
$((a_0,a_1),(b_0,b_1))$ by $[a_0,a_1] \times [b_0,b_1]$

IF $P \in \mathbb{P}^1 \times \mathbb{P}^1$ IS A POINT THEN WE DENOTE ITS IDEAL AS $\mathfrak{p}$ OR I_P OR $I(P)$

$$I_P = \{ F \in R \mid F(P) = 0 \} \subseteq R = k[x_0, x_1, y_0, y_1]$$

IF $P = [a_0,a_1] \times [b_0,b_1] \Rightarrow I(P) = (a_1 x_0 - a_0 x_1,\ b_1 y_0 - b_0 y_1)$

RECALL THE INTERPOLATION PROBLEM IN TERMS OF MULTIPROJECTIVE SPACES

PROBLEM 1
CLASSIFY THE NUMERICAL FUNCTIONS $H: \mathbb{N}^k \longrightarrow \mathbb{N}$
SUCH THAT $H = H_Z$ IS THE HILBERT FUNCTION
OF A MULTIGRADED RING R/I_Z WHERE

$I_Z = \mathfrak{p}_1^{m_1} \cap \dots \cap \mathfrak{p}_s^{m_s}$ FOR SOME SET

$X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ AND

$X = \{P_1, --, P_s\} \subseteq \mathbb{P}^{n_1} \times -- \times \mathbb{P}^{n_k}$ AND

MULTIPLICITIES $m_i \geqslant 1$

WE FOCUS ON SET OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$, SHOWING THE MAIN PROPERTIES OF THEIR HILBERT FUNCTION. WE ALSO A SHORT OVERVIEW IN CASE OF ARITHMETICALLY COHEN-MACAULAY (ACM FOR SHORT) SETS OF POINTS.

PROBLEM 1 CAN BE ANSWERED FOR REDUCED ACM SETS OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$, $m_i = 1$ AND WE SHOW SOME ADDITIONAL GEOMETRIC AND COMBINATORIAL PROPERTIES.

BIGRADED RINGS - PRELIMINARIES

LET $R = K[x_0, x_1, y_0, y_1]$ BE A POLYNOMIAL RING WITH COEFFICIENTS IN k, WHERE k IS AN ALGEBRAICALLY CLOSED FIELD OF $CHAR(k) = 0$

LET $\mathbb{N} = \{0, 1, 2, --\}$ BE THE SET OF NON-NEGATIVE INTEGERS, AND $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.
WE DENOTE BY $\preceq$ THE NATURAL PARTIAL ORDER ON $\mathbb{N}^2$ DEFINED BY

$$(a,b) \preceq (c,d) \iff \begin{matrix} a \leq c \text{ AND} \\ b \leq d \end{matrix}$$

LET $\underline{m} := (x_0, x_1, y_0, y_1)$ AND SET

$$\deg x_0 = \deg x_1 = (1,0)$$
$$\deg y_0 = \deg y_1 = (0,1)$$

A MONOMIAL $m = x_0^a x_1^b y_0^c y_1^d \in R$ HAS BIDEGREE

$$\deg m = (a+b, c+d)$$

FOR EACH $(i,j) \in \mathbb{N}^2$, WE DENOTE BY $R_{i,j}$ THE k-VECTOR SPACE SPANNED BY ALL THE MONOMIALS OF DEGREE (i,j)
THEN R IS A BIGRADED RING WITH DECOMPOSITION

$$R = \bigoplus_{(i,j) \in \mathbb{N}^2} R_{i,j}$$

SUCH THAT $R_{i,j} R_{k,l} \subseteq R_{i+k, j+l} \quad \forall (i,j), (k,l) \in \mathbb{N}^2$

SUCH THAT $R_{i,j} R_{k,l} \subseteq R_{i+k,j+l} \quad \forall \, (i,j),(k,l) \in \mathbb{N}^2$

$F \in R$ IS BIHOMOGENEOUS IF $F \in R_{i,j}$ FOR SOME $(i,j) \in \mathbb{N}^2$ AND $\deg F = (i,j)$

EX $F = x_0^3 y_0 y_1 + x_0 x_1^2 y_0^2$ HAS BIDEGREE $(3,2)$

BIHOMOGENEUS IDEALS

IF $I = (F_1, \dots, F_t) \subset R$ IS AN IDEAL WHERE EACH F_i IS BIHOMOGENEOUS $\Rightarrow I$ IS A BIHOMOGENEOUS IDEAL.

IF $I \subseteq R$ IS A BIHOMOG. IDEAL THEN WE

SET $I_{i,j} = I \cap R_{i,j}$ AND EACH $I_{i,j}$ IS A SUBVECTOR SPACE OF $R_{i,j}$

IN PARTICULAR, IF I IS BIHOMOG THEN

$$I = \bigoplus_{(i,j) \in \mathbb{N}^2} I_{i,j}$$

AND $(R/I)_{(i,j)} : R_{i,j} / I_{i,j} \quad \forall \, (i,j) \in \mathbb{N}^2$

$$\Rightarrow \quad R/I = \bigoplus_{(i,j) \in \mathbb{N}^2} (R/I)_{i,j}$$

DEFINITION 1

LET I BE A BIHOMOG IDEAL OF R. THE HILBERT FUNCTION OF R/I IS THE NUMERICAL FUNCTION

$$H_{R/I} : \mathbb{N}^2 \to \mathbb{N}$$

DEFINED BY
$$H_{R/I}(i,j) = \dim_K R_{i,j} - \dim I_{i,j}$$

WHERE $H_{R/I}(0,0) = 1$

WE WRITE THE OUTPUT OF THE HF AS AN INFINITE MATRIX OF TYPE

$$H_{R/I} \quad \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \; \cdots \\ \hline 0 & 1 & 2 & 3 & \cdots & \\ 1 & 2 & \ddots & & & \\ 2 & \vdots & & \ddots & & \\ 3 & \vdots & & & \ddots & \\ \vdots & & & & & \end{array} \quad \to \text{DEGREES IN } j$$

DEGREES IN i →

EX IF $I=(0) \Rightarrow H_{R/I}(i,j)=\dim R_{i,j} =$

$= (i+1)(j+1)$

SINCE THERE ARE $(i+1)(j+1)$ MONOMIALS IN BIDEGREE (i,j)

EX IF $I=(x_0-x_1,\ y_0-y_1)$ THEN

$$H_{R/I} \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 \rightarrow \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \rightarrow \\ 2 & \vdots & & \ddots & & & \\ 3 & \vdots & & & & & \\ 4 & \vdots & & & & & \end{array}$$

NOTE THAT I IS THE IDEAL OF JUST ONE POINT!

$P=[1,1]\times[1,1]$

ONE INTERESTING EXAMPLE IS GIVEN BY THE FOLLOWING

EX2 $I=(x_0,y_0)\cap(x_1,y_1)=$

$=(x_0x_1,\ x_0y_1,\ x_1y_0,\ y_0y_1)$

WE HAVE $\dim I_{0,0}=\dim I_{1,0}=\dim I_{0,1}=0$

I HAS MIN. GENS OF BIDEGREE $(2,0),(1,1),(0,2)$

THEN

$$H_{R/I} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 2 & \rightarrow \\ 1 & 2 & 2 & 2 & \rightarrow \\ 2 & \downarrow & \searrow & & \\ 3 & & & & \end{array}$$

WE ALSO NOTE THAT $V(I)=\Big\{\underbrace{[1,0]\times[1,0]}_{V(x_1,y_1)},\ \underbrace{[0,1]\times[0,1]}_{V(x_0,y_0)}\Big\}$

AND THAT I IS THE IDEAL OF TWO POINTS WHOSE COORDINATES ARE $[1,0]\times[1,0]$, AND $[0,1]\times[0,1]$ RESPECTIVELY

WE WILL SEE THAT THIS EXAMPLE IS THE CASE OF TWO NON ACM POINTS IN $\mathbb{P}^1\times\mathbb{P}^1$

DEF 2

LET $H:\mathbb{N}^2\rightarrow\mathbb{N}$ BE A NUMERICAL FUNCTION

THE FIRST DIFFERENCE OF H IS

$$\Delta H:\mathbb{N}^2\rightarrow\mathbb{N}$$

DEFINED BY

$$\Delta H(i,j) = H(i,j) - H(i-1,j) + H(i-1,j-1) - H(i,j-1)$$

WHERE $H(i,j) = 0$ IF $(i,j) \not\geq (0,0)$

EX3 SO FROM EX 1

$$\Delta H_{R/I} \quad \begin{array}{c|cccc} & 0 & 1 & & \\ \hline 0 & 1 & 0 & 0 & \rightarrow \\ 1 & 0 & \searrow & & \\ 1 & \downarrow & & & \end{array}$$

$I_X = (x_0 - x_1, y_0 - y_1) = \mathfrak{p}$ where

$P = [a_0, a_1] \times [b_0, b_1] =$
$[1,1] \times [1,1]$

EX2

$$\Delta H_{R/I} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 \end{array}$$

$X = \{P_1, P_2\}$,

where $I_{P_1} = (x_0, y_0)$ $P_1 [0,1] \times [0,1]$
$I_{P_2} = (x_1, y_1)$ $P_2 = [1,0] \times [1,0]$

WE NOTE THAT THE FIRST ΔH HAS ONLY 0 AND 1'S VALUES
WHILE THE SECOND ΔH HAS NEGATIVE VALUES
LET'S EXPLAIN THIS PROPERTY

RECALL

DEF IF $I \subset R$ IS A BIHOMOG IDEAL, THEN THE DEPTH OF I depth(I), is THE LENGTH OF THE MAXIMAL REGULAR SEQUENCE MODULO I

DEF THE KRULL DIMENSION OF R/I IS

$$K\text{-dim}(R/I) = \sup \left\{ ht_{R/I}(\mathfrak{p}) \;\middle|\; \mathfrak{p} \text{ IS A PRIME IDEAL OF } R/I \right\}$$

AND $ht_{R/I}(\mathfrak{p})$ IS THE LARGEST INTEGER t SUCH THAT

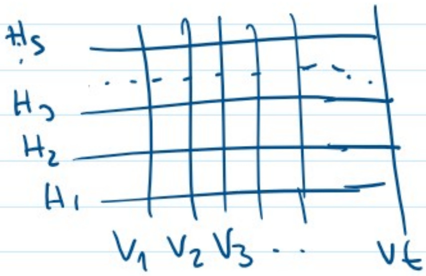
THERE EXIST PRIME IDEALS $\mathfrak{p}_i$ OF R/I : $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_t = \mathfrak{p}$

ARITHMETICALLY COHEN-MACAULAY IDEALS (ACM)

IF I_Y IS THE IDEAL ASSOCIATED TO A SET OF POINTS Y IN $\mathbb{P}^1 \times \mathbb{P}^1$, WE SAY THAT Y IS ACM IF R/I_Y IS COHEN-MACAULAY (DEPTH(R/I) = K-DIM(R/I))

COMBINATORIAL DESCRIPTION OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

IT'S KNOWN THAT A SMOOTH IRREDUCIBLE QUADRIC $Q \subset \mathbb{P}^3$ IS ISOMORPHIC TO $\mathbb{P}^1 \times \mathbb{P}^1$
AND IN $\mathbb{P}^1 \times \mathbb{P}^1$ WE HAVE TWO FAMILIES OF LINES $\{H_c\}$ AND $\{V_c\}$ EACH PARAMETRIZED BY $c \in \mathbb{P}^1$
WITH THE PROPERTY THAT LINES OF THE SAME FAMILY ARE SKEW AND LINES OF DIFFERENT FAMILY INTERSECT IN A POINT, THAT IS, A POINT $P = [a_0, a_1] \times [b_0, b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ CAN BE VIEWED AS THE INTERSECTION OF THE HORIZONTAL RULING DEFINED BY THE DEGREE $(1,0)$ LINE $H = a_1 x_0 - a_0 x_1$ AND THE VERTICAL RULING DEFINED BY THE DEGREE $(0,1)$ LINE $V = b_1 y_0 - b_0 y_1$
HENCE A FINITE SET OF POINTS X in $\mathbb{P}^1 \times \mathbb{P}^1$ CAN BE VIEWED AS



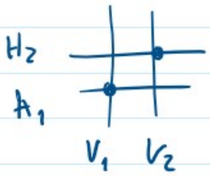
WE ASSOCIATE A LATTICE L_X OF ALL PAIRS (i,j) SUCH THAT $X \cap H_i \cap V_j \neq \emptyset$

LET'S GO BACK TO OUR PREVIOUS EXAMPLES

EX 1 $\quad X = \{P\} \quad P = [1,1] \times [1,1]$

$I_P = (\underset{H_1}{x_0 - x_1}, \underset{V_1}{y_0 - y_1})$

EX 2 $\quad X = \{P_1, P_2\} \quad P_1 = [0,1] \times [0,1]$
$P_2 = [1,0] \times [1,0]$



$H_1 \; x_0 = 0 \qquad V_1 \rightarrow y_0 = 0 \qquad I_{P_1}(x_0, y_0)$
$H_2 \; x_1 = 0 \qquad V_2 \rightarrow y_1 = 0 \qquad I_{P_2} = (x_1, y_1)$

DEFINITION

LET $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ BE THE PROJECTION MAP ONTO THE FIRST COORDINATE AND $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ ONTO THE

LET $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ BE THE PROJECTION MAP ONTO THE FIRST COORDINATE AND $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ ONTO THE SECOND COORDINATE.

LET $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ BE A FINITE SET OF REDUCED POINTS AND SUPPOSE $\pi_1(X) = \{A_1, \dots, A_h\}$ AND $\pi_2(X) = \{B_1, \dots, B_v\}$

FOR $i = 1, \dots, h$ SET $\alpha_i := |\pi_1^{-1}(A_i) \cap X|$ AND

$\alpha_X := (\alpha_1, \dots, \alpha_h)$

FOR $j = 1, \dots, v$ SET $\beta_j := |\pi_2^{-1}(B_j) \cap X|$ AND

$\beta_X := (\beta_1, \dots, \beta_v)$

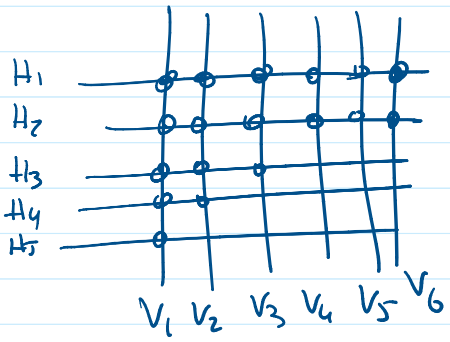
DEFINITION

GIVEN $\alpha_X := (\alpha_1, \dots, \alpha_h)$ AND $\beta_X := (\beta_1, \dots, \beta_v)$ THE CONJUGATE OF α_X IS THE TUPLE

$$\alpha_X^* = (\alpha_1^*, \dots, \alpha_{\alpha_1}^*)$$

WHERE $\alpha_j^* = |\{\alpha_i \in \alpha_X : \alpha_i \geq j\}|$

EXAMPLE Let X be A SET OF POINTS AN IN FIGURE



$\Rightarrow \alpha_X = (6, 6, 3, 2, 1)$

OR $\beta_X = (5, 4, 3, 2, 2, 2)$

THE NUMBER α_i COUNTS THE NUMBER OF POINTS IN X WHOSE FIRST COORDINATE IS A_i (RECALL $\pi_1(X) = \{A_1, \dots, A_h\}$)

WE HAVE THE CONVENTION THAT $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_h$.
ANALOGOUSLY, WE CAN DO THE SAME WITH β_X

DEF FOR ANY PARTITION $\lambda = (\lambda_1, \dots, \lambda_r) \longmapsto s$

WE ASSOCIATE AN $(r \times \lambda_1)$ GRID PLACING λ_1 POINTS ON H_1. λ_2 POINTS ON H_2. ECT.

WE ASSOCIATE AN $(\ell \times \lambda_1)$ GRID PLACING λ_1 POINTS ON H_1, λ_2 POINTS ON H_2, ECT.
THE RESULTING DIAGRAM IS CALLED FERRERS DIAGRAM OF λ AND X RESEMBLES A FERRERS DIAGRAM

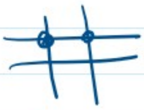
EXAMPLE LET X BE WITH
ASSOCIATED $\alpha_X = (6,6,3,2,1)$

WE NOTE THAT $\alpha^*_X = (5,4,3,2,2,2) = \beta_X$

EXAMPLE LET X BE  WE HAVE

$\alpha_X = (1,1)$ AND $\beta_X = (1,1)$. $H_X = \begin{array}{|llll} \hline 1 & 2 & 2 & 2 \to \\ 2 & 2 & 2 & 2 \\ \downarrow & & \searrow & \end{array}$

$\alpha^*_X = (2,0)$. HENCE $\alpha^*_X \neq \beta_X$

EXAMPLE LET X be 

$\alpha_X = (2,0)$ AND $\beta_X = (1,1)$ $H_X = \begin{array}{|lll} \hline 1 & 2 & 2 \to \\ 1 & 2 & 2 \to \\ \downarrow & & \end{array}$

$\alpha^*_X = (1,1)$. HENCE $\alpha^*_X = \beta_X$

WE HAVE SEEN TWO EXAMPLE WHERE IT IS NOT ALWAYS TRUE THAT $\alpha^*_X = \beta_X$!

FROM [G-VT] WE CAN PROVE THAT

THM $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ BE A SET OF POINTS THAT RESEMBLES A FERRERS DIAGRAM $\Rightarrow \alpha^*_X = \beta_X$ (AND $\beta^*_X = \alpha_X$)

WE INTRODUCE AN OTHER CONSTRUCTION

LET X BE A FINITE SET OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$ WITH

$H_1, \ldots, H_h$ HORIZONTAL LINES AND $V_1, \ldots, V_v$

VERTICAL LINES.

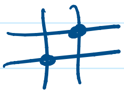
FOR $i = 1, \ldots, h$, LET $S_i = (S_{i,1}, \ldots, S_{i,v}) \in \mathbb{N}^v$ BE

A V-TUPLE WHERE $S_{i,j} = 1$ IF $H_i \cap H_j \in X$, AND

0 OTHERWISE.

A v-TUPLE WHERE $s_{ij}=1$ IF $H_i \cap H_j \in X$, AND 0 OTHERWISE.

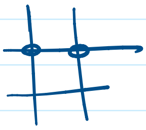
SET $\mathcal{S}_X = \{s_1, \dots, s_h\}$

<u>EXAMPLE</u> LET X BE 

$\Rightarrow \begin{array}{l} s_1 = (0,1) \\ s_2 = (1,0) \end{array} \Rightarrow \mathcal{S}_X = \{(0,1),(1,0)\}$

WE NOTE THAT, ACCORDING TO OUR ORDERING, WE HAVE

$s_1 \not\geq s_2 \Rightarrow \mathcal{S}_X$ IS NOT TOTALLY ORDERED

<u>EXAMPLE</u> LET X BE 

$s_1 = (1,1)$ $\mathcal{S}_X = (1,1)$. TRIVIALLY, $\mathcal{S}_X$ IS T.O.

<u>EXAMPLE</u> X with $\alpha = (6,6,3,2,1)$

$s_1 = (1,1,1,1,1,1)$
$s_2 = (1\ 1\ 1\ 1\ 1\ 1)$
$s_3 = (1\,1\,1\ 0\ 0\ 0)$ $\Rightarrow \mathcal{S}_X$ IS T.O.
$s_4 = (1\,1\ 0\ 0\ 0\ 0)$
$s_5 = (1\ 0\,0\,0\,0\,0$

WE CAN PROVE THAT

THM [G-VT, THM 3.21] LET $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ BE A REDUCED SET OF POINTS. THE FOLLOWING ARE EQUIVALENT:

① X IS ACM

② $\mathcal{S}_X$ is t.o.

③ $\alpha^*_X = \beta_X$

④ THE FIRST DIFFERENCE FUNCTION ΔH_X is the BIGRADED HILB. FUNCTION OF AN ARTINIAN QUOTIENT OF $K[x_1, y_1]$

QUOTIENT OF $K[x_1, y_1]$

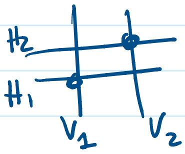
⑤ X SATISFIES THE $(*)$-PROPERTY, THAT IS
IF WHENEVER $A \times B$ ARE IN X AND $A' \times B'$ ARE IN X
WITH $A \neq A'$ AND $B \neq B'$, THEN EITHER $A \times B'$ OR
$A' \times B$ (OR BOTH) IS ALSO IN X.

⑥ $\forall \; P \in X \Rightarrow |\deg_X P| = 1$

WHERE THE DEGREE OF A POINT $P \in X$ is THE SET
$\deg_X P = \min \{ \deg F \mid F$ IS A SEPARATOR FOR P in $X \}$

PROOF SEE [G-VT], THEOR 4.11]

EX LET X BE

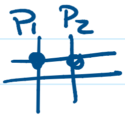


$P_1 = H_1 \cap V_1$

CONSIDER $P_1 = H_1 \cap V_1$

WE HAVE THAT THE FORMS H_2 AND V_2 ARE SEPARATORS
FOR $P_1 \Rightarrow |\deg_X(P)| > 1$
AND WE HAVE SEEN $\alpha_X^* \neq \beta_X$, ΔH_X HAS NEGATIVE
VALUES AND $|\deg_X(P)| > 1$, $S_X = \{(0,1), (1,0)\}$ NOT T.O.
X IS NOT ACM

EX LET X BE



$|\deg_X(P_1)| = 1$

$|\deg_X(P_2)| = 1$

X IS ACM

$\alpha_X^* = \beta_X$, $\Delta H_X \; \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & & & & \end{array}$, $S_X = \{(1,1)\}$

EX LET X BE $\alpha_X = (6,6,3,2,1)$

S_X is t.o. , $\alpha^*_X = \beta_X$

ΔH	0	1	2	3	4	5	6
0	1	1	1	1	1	1	0
1	1	1	1	1	1	1	0
2	1	1	1	0	0	0	0
3	1	1	0	0	0	0	0
4	1	0	0	0	0	0	0
5	0	0	0	0	0	0	

$\Longleftrightarrow$ RECALL [GVT] IN EACH ROW i HAS AS MANY 1's AS THE NUMBER OF POINTS IN EACH LINE H_i

8.3 Tutorial Problems

Here are the associated tutorial problems.

HILBERT FUNCTIONS OF POINTS (SMS COMMUATATIVE ALGEBRA SCHOOL)
ELENA GUARDO AND ADAM VAN TUYL

TUTORIAL 1

The goal of this tutorial is to help you become more familiar with the basic properties of ideals of points, Hilbert functions, and the Hilbert functions of ideals of points in $\mathbb{P}^2$. The problems do not have to be done in any particular order. Tackle the ones that interest you the most. You may want to start with the last section of this tutorial for some hints on how to do computations involving points and Hilbert functions in Macaulay2 (also see the lectures of F. Galetto).

I. THE IDEAL OF A SET OF POINTS. The first collection of questions focuses on properties of the ideal of a set of points. These questions highlight some of the algebraic features of these ideals.

Exercise 1. We defined $\mathbb{P}^n$ to be the set $\mathbb{K}^{n+1} \setminus \{(0,\ldots,0)\}$ with an equivalence relation defined by:

$$(a_0,\ldots,a_n) \sim (b_0,\ldots,b_n) \text{ if and only if there is } 0 \neq c \in \mathbb{K} \text{ such that } (b_0,\ldots,b_n) = (ca_0,\ldots,ca_n).$$

Convince yourself that this is an equivalence relation.

Exercise 2. For any point $P \in \mathbb{P}^n$, verify that $I(P)$ is a homogeneous ideal.

Exercise 3. Let $P = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ with $a_0 \neq 0$. In the lecture we stated

$$I(P) = \langle a_1x_0 - a_0x_1, a_2x_0 - a_0x_2, \ldots, a_nx_0 - a_0x_n \rangle.$$

Prove this claim. Find generators for the ideal $I(P)$ if $a_0 = 0$.

Hint. Note that the containment $\supseteq$ is straightforward. For the other direction, apply the generalized division algorithm with a monomial order $x_n > x_{n-1} > \cdots > x_0$.

Exercise 4. If $P \in \mathbb{P}^n$, what is a Gröbner basis of $I(P)$?

Exercise 5. For any point $P \in \mathbb{P}^n$, prove

(1) $I(P)$ is a prime ideal;
(2) $I(P)$ is a complete intersection; and
(3) $\dim R/I(P) = \operatorname{depth} R/I(P) = 1$ where $R = \mathbb{K}[x_0,\ldots,x_n]$.

Hint. Try the special case $P = [1 : 0 : 0 : \cdots : 0]$ first.

Exercise 6. For any finite set of points $\mathbb{X} \subseteq \mathbb{P}^n$, prove

(1) $\sqrt{I_{\mathbb{X}}} = I_{\mathbb{X}}$; and
(2) $\dim R/I_{\mathbb{X}} = \operatorname{depth} R/I_{\mathbb{X}} = 1$ where $R = \mathbb{K}[x_0,\ldots,x_n]$.

Exercise 7. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a finite set of points. Prove that $I_{\mathbb{X}}$ is a prime ideal if and only if $\mathbb{X} = \{P\}$.

Exercise 8.

(1) Let $\mathbb{X} = \{[1:0:0],[0:1:0],[0:0:1]\} \subseteq \mathbb{P}^2$. Prove that $I_{\mathbb{X}}$ is a monomial ideal. Is this ideal a complete intersection?
(2) Let $\mathbb{X} = \{[1:0:0],[0:1:0],[0:0:1],[1:1:1]\} \subseteq \mathbb{P}^2$. Prove that $I_{\mathbb{X}}$ is a complete intersection.

Remark. Note that Exercise 8 (2) shows that the converse of Exercise 5 (2) is false, that is, there are sets of points that are complete intersections that are not a single point.

Exercise 9. Find sets of points $\mathbb{X} \subseteq \mathbb{P}^n$ such that $I_{\mathbb{X}}$ is a monomial ideal.

II. HILBERT FUNCTIONS AND MACAULAY'S THEOREM. The second collection of questions focuses on understanding Hilbert functions and the statement of Macaulay's Theorem.

Exercise 10. Let $R = \mathbb{K}[x_0, \ldots, x_n]$ and let I be a homogeneous ideal of R. Recall that for any integer $d \geq 0$, we define

$$R_d = \{f \in R \mid f \text{ is homogeneous of degree } d\}$$

and

$$I_d = \{f \in I \mid f \text{ is homogeneous of degree } d\}.$$

Verify that R_d and I_d are $\mathbb{K}$-vector spaces, and that I_d is a subspace of R_d.

Exercise 11. Let $R = \mathbb{K}[x_0, x_1, x_2]$ and compute $\dim_{\mathbb{K}} R_d$ for $d = 0, \ldots, 10$. Where do these values appear in Pascal's triangle? Repeat for $R = \mathbb{K}[x_0, x_1, x_2, x_3]$.

Exercise 12. Let $a = 2025$. Compute $a^{\langle i \rangle}$ for $i = 2, 100$, and 2025.

Exercise 13. Prove that $H: \ 1 \ 3 \ 6 \ 10 \ 15 \ \cdots \ \binom{i+2}{2} \ \cdots$ is an O-sequence.

Exercise 14. Which of following sequences are valid Hilbert functions? If the sequence is valid, can you find a polynomial ring R and homogeneous ideal I such the Hilbert function of R/I is given by the sequence?

(1) $H_1: \ 1 \ 3 \ 6 \ 10 \ 9 \ 20 \ 30 \ 40$ (increasing by 10)
(2) $H_2: \ 1 \ 3 \ 6 \ 10 \ 9 \ 9 \ 9 \rightarrow$ (stabilizes at 9)
(3) $H_3: \ 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \rightarrow$ (stabilizes at 9)

Exercise 15. Let F be a homogeneous element of $R = \mathbb{K}[x_0, \ldots, x_n]$ of degree d. If $I = \langle F \rangle$, prove that the Hilbert function of R/I is given by

$$H_{R/I}(i) = \begin{cases} \binom{i+n}{n} & \text{if } 0 \leq i < d \\ \binom{i+n}{n} - \binom{i-d+n}{n} & \text{if } d \leq i. \end{cases}$$

III. HILBERT FUNCTIONS OF SETS OF POINTS. The third collection of questions focuses on the properties of Hilbert functions of sets of points.

Exercise 16. Let $\mathbb{X}$ be any set of s points in $\mathbb{P}^1$. Prove that the Hilbert function of $R/I_{\mathbb{X}}$ is

$$H_{R/I_{\mathbb{X}}}: \ 1 \ 2 \ 3 \ 4 \cdots \ s-1 \ s \ s \rightarrow .$$

Exercise 17. Let $\mathbb{X}$ be any set of s points *on a line* in $\mathbb{P}^n$. Prove that the Hilbert function of $R/I_{\mathbb{X}}$ is

$$H_{R/I_{\mathbb{X}}}: \ 1 \ 2 \ 3 \ 4 \cdots \ s-1 \ s \ s \rightarrow .$$

Exercise 18. Write out all the possible Hilbert functions of five points in $\mathbb{P}^2$. Repeat for ten points in $\mathbb{P}^3$.

Exercise 19. Prove that any set of five points in $\mathbb{P}^2$ lie on a quadric.

Remark. The question is asking you to show that regardless of how you pick the five points, there is a homogeneous element of degree two in $\mathbb{K}[x_0, x_1, x_2]$ that vanishes at all the points. How does the previous question help?

Exercise 20. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be any finite set of points. Prove that $H_{R/I_X}(i) \leq H_{R/I_X}(i+1)$ for all $i \geq 0$.

Hint. Recall that there is a non-zero divisor $\overline{L} \in R/I_{\mathbb{X}}$. We then have a short exact sequence with degree 0 maps:

$$0 \longrightarrow R/I_{\mathbb{X}}(-1) \xrightarrow{\times \overline{L}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, L) \longrightarrow 0.$$

Exercise 21. In the lecture, it was stated that H is the Hilbert function of a set of points in $\mathbb{P}^2$ if and only if there exists integers α and σ such that

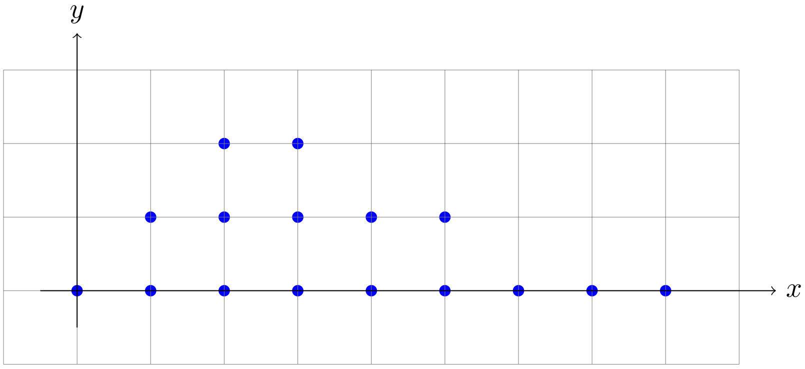
(1) $\Delta H(i) = i + 1$ for $0 \leq i < \alpha$
(2) $\Delta H(i + i) \leq \Delta H(i)$ for $\alpha \leq i < \sigma$
(3) $\Delta H(i) = 0$ for $\sigma \leq i$.

Convince yourself that $H : 1\ 3\ 6\ 9\ 11\ 13\ 14\ 15\ 16\ 16 \rightarrow$ is a valid Hilbert function of a set of points in $\mathbb{P}^2$. What is the α and σ for this sequence?

Exercise 22. In the previous problem, you showed that $H : 1\ 3\ 6\ 9\ 11\ 13\ 14\ 15\ 16\ 16 \rightarrow$ is the Hilbert function of a set of points in $\mathbb{P}^2$. This only tells us that there is a set of points in $\mathbb{P}^2$ with this Hilbert function, but it doesn't tell us how to construct a set of points with this Hilbert function.

Here is one procedure to construct a set of points with this Hilbert function.

(1) From H, determine the sequence ΔH. For example, using the sequence above, we have $\Delta H : 1\ 2\ 3\ 3\ 2\ 2\ 1\ 1\ 1\ 0$.
(2) We use ΔH to make a plot of points. Specifically, for each integer $i \geq 0$, we plot the points $\{(i, 0), (i, 1), \ldots, (i, \Delta H(i))\}$. Keep in mind that our Hilbert function starts at $i = 0$. For example, using our ΔH as above, we have the following graph[1]



(3) We now "projectivize" the plotted points, i.e., point (a, b) in the above grid becomes $[1 : a : b] \in \mathbb{P}^2$. In our example, we have

$$\mathbb{X} = \{[1:0:0], [1:1:0], [1:1:1], [1:2:0], [1:2:1], [1:2:2], [1:3:0], [1:3:1], [1:3:2], [1:4:0], [1:4:1], [1:5:0], [1:5:1], [1:6:0], [1:7:0], [1:8:0]\}.$$

Verify that the ideal of these 16 points has the desired Hilbert function.

Exercise 23. Prove that the above procedure always gives a sets of points with the correct Hilbert function.

Remark. You may wish to consult the original paper of Geramita, Maroscia, and Roberts [1].

---

[1] Graph courtesy of ChatGPT

Exercise 24. Find a set of points in $\mathbb{P}^2$ with Hilbert function $H : 1\ 3\ 6\ 10\ 15\ 21\ 21 \rightarrow$

Exercise 25. For each Hilbert function you found in Exercise 18, find a set of points with that Hilbert function.

Exercise 26. Let $R = \mathbb{K}[x_0, x_1, x_2]$. Let L_1, L_2, L_3, L_4, L_5 be a general set of degree one forms of R. That is, each $L_i \in R_1$, and any three of them are linearly independent, i.e., they form a basis of the vector space R_1. Geometrically, each L_i corresponds to a line ℓ_i in $\mathbb{P}^2$, and the ideal $\langle L_i, L_j \rangle$ is the ideal of the point $P_{i,j} = \ell_i \cap \ell_j$. Consider the ideal

$$I_{\mathbb{X}} = \bigcap_{1 \leq i < j \leq 5} \langle L_i, L_j \rangle.$$

Prove that $I_{\mathbb{X}}$ is the ideal of 10 points in $\mathbb{P}^2$ whose Hilbert function is $H :\ 1\ 3\ 6\ 10\ 10 \rightarrow$.

Remark. The set $\mathbb{X}$ in the previous question is called a *star configuration* since the five general lines look like a star as shown in Figure 1. For more on star configurations, see [2].

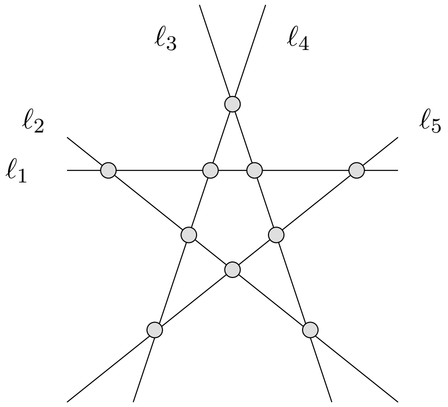


FIGURE 1. A star configuration of 10 points in $\mathbb{P}^2$

Exercise 27. Repeat the previous question, but use 7 general lines instead of 5. How many points do you construct? What is its Hilbert function? Instead of 7 lines, use t lines for some integer $t \geq 7$. How many points do you construct? What is its Hilbert function?

Exercise 28. Consider a "random" set of 21 points in $\mathbb{P}^2$. Try as many different random sets as you can. What do you expect the Hilbert function to be? Instead of $s = 21$, try another number. Is there an expected Hilbert function? Can you prove your guess?

Remark. This question is more open-ended. You are being asked to run some computer experiments to guess what "most" Hilbert functions of ideals sets of points should look like. Some of the code in the next section may help.

Exercise 29. In $\mathbb{P}^n$, consider the following $n + 1$ points:

$$\mathbb{X} = \{[1 : 0 : \cdots : 0],\ [0 : 1 : 0 : \cdots : 0], \ldots, [0 : 0 : \cdots : 0 : 1]\}.$$

(1) Prove that $I_{\mathbb{X}}$ is a monomial ideal.
(2) Find the Hilbert function of $R/I_{\mathbb{X}}$ where $R = \mathbb{K}[x_0, \ldots, x_n]$.
(3) What is the simplicial complex associated to $I_{\mathbb{X}}$ via the Stanley-Reisner correspondence?

(4) What is the f-vector and h-vector of the simplicial complex from the previous part? How does this relate to the Hilbert function of $R/I_{\mathbb{X}}$?

Hint. You may want to try $n = 2$ first.

IV. HILBERT FUNCTIONS, POINTS, AND MACAULAY2. The last collection of problems deal with computing some of the objects discussed in the lecture.

Exercise 30. Create a Macaulay2 function that will take as input a point $P = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ and return the homogeneous ideal $I(P)$.

Answer. We provide an answer to this question in case you simply want to use the code for the other problems. This code is **not** optimal! It is provided simply to give you something with which to work.

```
idealPoints = P -> (
n = numgens(R);
gensIdeal = {};
for i from 0 to (n-1) do (
    gensIdeal = append(gensIdeal,P_0*x_i-P_i*x_0);
    );
i = ideal( mingens ideal(gensIdeal));
return i;
);
```

Here is an example of the code:

```
i01 : R = QQ[x_0..x_4]
o01 = R
o01 : PolynomialRing

i02 : P = {-1,2,2025,3,17}
o02 = {-1,2,2025,3,17}
o02 : List

i03 : idealPoints(P)
o03 = ideal(17x_3 - 3x_4,17x_2 - 2025x_4, 17x_1 - 2x_4,17x_0 + x_4)
o03 : Ideal of R
```

Exercise 31. Create a function that takes a list of points in $\mathbb{P}^n$ as input and returns the defining ideal of the set of points.

Exercise 32. Suppose that I is a homogeneous ideal of R. In Macaulay2, the value of the Hilbert function $H_{R/I}(d)$ can be accessed using the command `hilbertFunction(d,I)`. For example

```
i1 : R=QQ[x_0,x_1,x_2];
i2 : I = idealPoints({1,2,3});
i3 : hilbertFunction(2,I)
o3 : 1
```

This tells us that the Hilbert function of R/I in degree 2 is 1. Write a function that inputs an ideal and returns the first 10 values of the Hilbert function. Adapt your code so that the user can determine the number of values of the Hilbert function that are returned.

Exercise 33. Review the documentation for `macaulayExpansion(a,i)`. Use Macaulay2 to check your answer to Exercise 12.

Exercise 34. Write a program that takes as input positive integers n and s and returns all valid Hilbert functions of s points in $\mathbb{P}^n$.

Exercise 35. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a set of s points. Can you find a relationship between the Castelnuovo-Mumford regularity of $I_{\mathbb{X}}$ and the Hilbert function of $R/I_{\mathbb{X}}$? Use Macaulay2 to make your conjecture. In the case of $\mathbb{P}^2$, relate your answer to the σ in Exercise 21.

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[1] A. V. Geramita, P. Maroscia and L. G. Roberts, The Hilbert function of a reduced k-algebra, J. London Math. Soc. (2) **28** (1983), no. 3, 443–452; MR0724713

[2] A. V. Geramita, B. Harbourne and J. C. Migliore, Star configurations in $\mathbb{P}^n$, J. Algebra **376** (2013), 279–299; MR3003727

HILBERT FUNCTIONS OF POINTS (SMS COMMUATATIVE ALGEBRA SCHOOL)
ELENA GUARDO AND ADAM VAN TUYL

TUTORIAL 2

The following problems are based upon the second and third lecture on the Hilbert functions of points. The problems do not have to be done in any particular order. Tackle the ones that interest you the most. You may want to look at the end of this tutorial for some hints on how to do computations involving points in Macaulay2.

I. SET OF FAT POINTS AND THEIR HILBERT FUNCTIONS**.** The first collection of questions focuses on properties of the ideal of a set of fat points. We highlight some of the algebraic features of these ideals.

We use the following notation for a set of fat points. Let $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ be a set of distinct reduced points, and $m_1, \ldots, m_s$ positive integers. We let $Z = m_1P_1 + \cdots + m_sP_s$ denote the scheme defined by the ideal

$$I_Z = I(P_1)^{m_1} \cap I(P_2)^{m_2} \cap \cdots \cap I(P_s)^{m_s}.$$

Exercise 1. Let $\mathbb{X} = \{[1:0:1], [1:2:3]\} = \{P_1, P_2\} \subseteq \mathbb{P}^2$. What is the defining ideal of the set of fat points $Z = 3P_1 + 2P_2$, i.e., what are the generators of this ideal?

Exercise 2. Prove that for any set of fat points $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$ there is a non-zero divisor $\overline{L} \in R/I_Z$ where L is a homogeneous element of degree 1.

Exercise 3. For any point $P \in \mathbb{P}^n$ and positive integer $m \geq 1$, prove
(1) $I(P)^m$ is a primary ideal;
(2) $I(P)^m$ is a complete intersection if and only if $m = 1$; and
(3) $\dim R/I(P)^m = \text{depth } R/I(P)^m = 1$ where $R = \mathbb{K}[x_0, \ldots, x_n]$.

Hint. Try the special case $P = [1:0:0:\cdots:0]$ first.

Exercise 4. For any set of points of fat points $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$, prove
(1) $\sqrt{I_Z} = I_{\mathbb{X}}$ where $\mathbb{X} = \{P_1, \ldots, P_s\}$. ; and
(2) $\dim R/I_Z = \text{depth} R/I_Z = 1$ where $R = \mathbb{K}[x_0, \ldots, x_n]$.

Exercise 5. Let $\mathbb{X} = \{P_1, P_2, P_3\} = \{[1:0:0], [0:1:0], [0:0:1]\} \subseteq \mathbb{P}^2$. Prove that the defining ideal of $Z = 2P_1 + 3P_2 + 4P_3$ is a monomial ideal.

Exercise 6. Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points of $\mathbb{P}^n$. Suppose that I_Z is a square-free monomial ideal. Prove that $m_1 = \cdots = m_s = 1$.

Exercise 7. Let $P \in \mathbb{P}^n$ and $m \geq 1$ for any postive integer m. What is the Hilbert function of the fat point $Z = mP$?

Hint. This problem is easier to solve if you assume $P = [1:0:\cdots:0]$. What is I_Z in this case?

Exercise 8. If $P \in \mathbb{P}^n$ and $m \geq 1$ is an integer, then the *degree* of the fat point mP is $\binom{m+n-1}{n}$. How is the degree of fat point related to the answer of your previous question.

Exercise 9. Let $Z \subseteq \mathbb{P}^n$ be any set of fat points. Prove that $H_{R/I_Z}(i) \leq H_{R/I_Z}(i+1)$ for all $i \geq 0$.

Exercise 10. Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$. We define $\deg Z = \sum_{i=1}^{s} \binom{m_i+n-1}{n}$. Prove that $H_{R/I_Z}(i) \leq \deg Z$ for all i.

Exercise 11. Explain why $H : 1\ 3\ 6\ 10\ 15\ 21\ 20\ 20 \rightarrow$ cannot be the Hilbert function of a set of fat points in $\mathbb{P}^2$. Show that $H : 1\ 2\ 3\ 3 \rightarrow$ is valid Hilbert function of distinct points in $\mathbb{P}^2$, but there is no set of fat points with this Hilbert function.

II. THE IDEAL OF A SET OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$**.** The next collection of questions focuses on properties of the ideal of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We highlight some of the algebraic features of these ideals.

The coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is $R = k[x_0, x_1, y_0, y_1]$ with a bigrading induced by $\deg x_0 = \deg x_1 = (1, 0)$ and $\deg y_0 = \deg y_1 = (0, 1)$. The (bi)degree of the monomial $x_0^{a_0} x_1^{a_1} y_0^{b_0} y_1^{b_1}$ is $(a_0 + a_1, b_0 + b_1)$. An element of $F \in R$ is *bihomogeneous* if every monomial term in F has the (bi)degree. An ideal is *bihomogeneous* if it is generated by bihomogeneous elements.

Exercise 12. For any point $P \in \mathbb{P}^1 \times \mathbb{P}^1$, verify that $I(P)$ is a bihomogeneous ideal.

Exercise 13. Let $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ be a point. Prove that

$$I(P) = \langle a_1x_0 - a_0a_1, b_1y_0 - b_0y_1 \rangle$$

Exercise 14. For any point $P \in \mathbb{P}^1 \times \mathbb{P}^1$, prove

(1) $I(P)$ is a prime ideal;
(2) $I(P)$ is a complete intersection; and
(3) $\dim R/I(P) = \text{depth } R/I(P) = 2$ where $R = \mathbb{K}[x_0, x_1, y_0, y_1]$.

Exercise 15. Prove that for any set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, there is a non-zero divisor $\overline{L} \in R/I_{\mathbb{X}}$ where L is a bihomogeneous element of degree $(1, 0)$. Prove that you can also find a non-zero divisor $\overline{L'}$ where $\deg L' = (0, 1)$.

Exercise 16. Prove that for any set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, $\dim R/I_X = 2$.

Exercise 17. Find a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\text{depth}(R/I_X) = 1$. Find another set of points $\mathbb{Y}$ with $\text{depth}(R/I_{\mathbb{Y}}) = 2$.

Remark. For points and fat points in $\mathbb{P}^n$, the coordinate ring is always Cohen-Macaulay since we always that the depth and dimensions of these rings are the same. However, the above exerice implies that there are sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are Cohen-Macualay, and some that are not.

Exercise 18. Let $R = k[x_0, x_1, y_0, y_1]$ with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. Let $R_{i,j}$ denote the vector space of all bihomegeneous elements of degree (i, j). What is $\dim_k R_{i,j}$. Use this result to find the bigraded Hilbert function of R

Exercise 19. How does the previous exercise change if $R = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$.

Exercise 20. Let $P \in \mathbb{P}^1 \times \mathbb{P}^1$. What is the bigraded Hilbert function of $R/I(P)$.

Exercise 21. Suppose that $X = \{P_1, P_2\}$ is a set of two distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$. Show that there are exactly three possible bigraded Hilbert functions for $R/I(X)$. Which of these three Hilbert functions correspond to an ACM set of points.

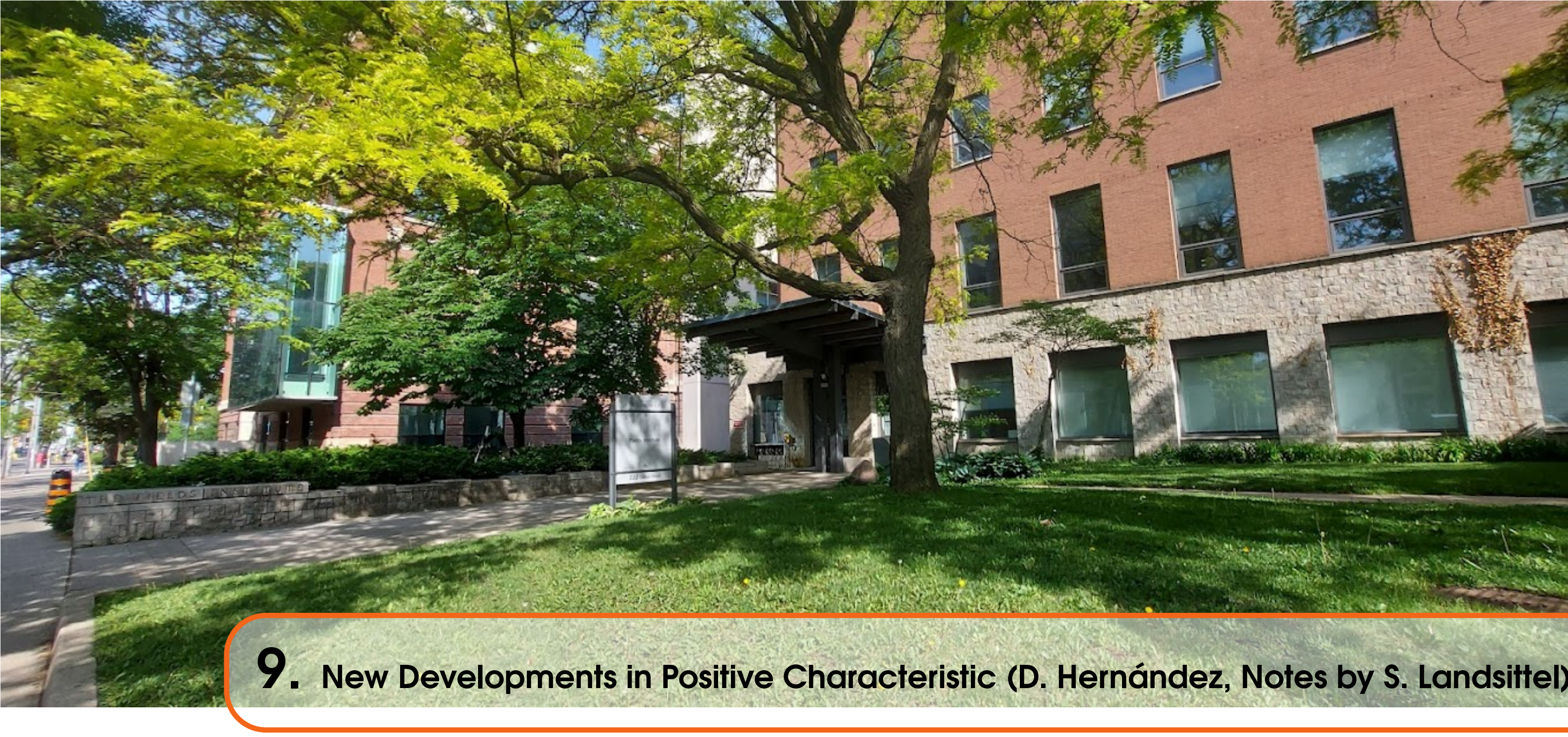
Exercise 22. Fix a point $A \in \mathbb{P}^1$ and let $\{B_1, \ldots, B_s\}$ be s distinct points in $\mathbb{P}^1$. Let $X = \{A \times B_1, A \times B_2, \ldots, A \times B_s\}$ be s points in $\mathbb{P}^1 \times \mathbb{P}^1$. What is the bigraded Hilbert function of $R/I(X)$? Determine if X is ACM.

Exercise 23. With X as in the last problem, what is bigraded minimal free resolution of of $I(X)$?

Exercise 24. How would you define a point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$?

Exercise 25. Find a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where $\dim R/I(X) = 4$, but $\operatorname{depth} R/I(X) = 2$.

Exercise 26. Let $P \in \mathbb{P}^1 \times \mathbb{P}^1$. We can define a "fat point" in $\mathbb{P}^1 \times \mathbb{P}^1$ similarly to points in $\mathbb{P}^n$. It is the scheme defined by $I(P)^m$ with $m \geq 1$ an integer. Determine a formula for the bigraded Hilbert function of $R/I(P)^m$



9. New Developments in Positive Characteristic (D. Hernández, Notes by S. Landsittel)

This course surveyed recent results on F-singularity theory, such as new results about F-regularity, F-pure thresholds, and test ideals. This course was be taught by Daniel Hernández (Kansas).

9.1 Lecture Notes

We have included copies of Daniel's lecture notes and his tutorials. The lecture notes were provided by Stephen Landsittel.

A NUMERICAL INVARIANT IN PRIME CHARACTERISTIC

These are notes taken (verbatim or paraphrased) from a series of three lectures by Daniel Hernández at the Fields Institute in Toronto Canada in June of 2025. Throughout these notes, p will be a (positive) prime integer, and q will denote various natural powers of p.

1. Preliminaries

Setup. Let k be a perfect field of prime characteristic p (so $k = k^p$) and let (R, m) be an $F-$finite regular local ring (e.g. we could take R to be $k[\underline{x}]_{(\underline{x})}$ or $k[[\underline{x}]]$).

Definition 1.1. We have a ring map $F : R \to R$ given by $r \mapsto r^p$ for $r \in R$ since p is the characteristic of R. F is called the *Frobenius Homomorphism.*

Definition 1.2. R has the subring

$$R^p := \{r^p \mid r \in R\}.$$

We see that $R^p \subset R$ is actually a subring using the fact that Frobenius $F : R \to R$ is a ring map.

Example 1.3. We can easily compute some examples of R^p for our aforementioned localized polynomial and power series rings as follows. We can compute these using the fact that

$$(k[\underline{x}])^p = k^p[\underline{x}^p] = k[\underline{x}^p] \cong k[\underline{x}]$$

as rings (since k is perfect). △

Definition 1.4. Since R is a domain, we can fix an algebraic closure L of the field of fractions of R, and we can look at the ring $R^{1/p}$, which is the following subring of L

$$R^{1/p} := \{r^{1/p} \in L \mid r \in R\}.$$

Comments 1.5. *By repeatedly constructing the rings of the preceding two definitions, we get a commutative diagram of rings*

$$\begin{tikzcd} ... \arrow[r, shift left] & R^{p^2} \arrow[l, shift left] \arrow[r, shift left] & R^p \arrow[l, shift left] \arrow[r, shift left] & R \arrow[l, shift left] \arrow[r, shift left] & R^{1/p} \arrow[l, shift left] \arrow[r, shift left] & R^{1/p^2} \arrow[l, shift left] \arrow[r, shift left] & ... \arrow[l, shift left] \end{tikzcd}$$

where we see a fractal-like behavior with an arbitrary starting point (at the ring R we picked earlier, we could have started at $R^{1/q}$ or R^q for some q for instance). All of the rings to the right of R in the above diagram are free R-modules by Kunz's Theorem (which we will state a version of shortly).

Remark 1.6. For all q, and ideals I of R the set $J := \{a^q \mid a \in I\}$ is an ideal in the ring R^q, and hence we get an ideal by extension of this ideal to the ring R

$$JR = \left\{ \sum_{i=1}^{s} r_i a_i^q \mid s \geq 1, r_i \in R, a_i \in I \right\}$$

which is denoted by $I^{[q]}$. The ideal $I^{[q]}$ of R is called the q^{th} bracket power of I.

Lemma 1.7. *For all $q = p^e$ and $f \in R$ we have that $f^q \in I^{[q]}$*

Theorem 1.8. *(Kunz) Since R is a regular local ring, we have that $R^{1/q}$ is a free R-module for all q.*

Some notes when $R = k[[x, y]]$

It suffices look at the case when when $q = p$. We have that $R^p \subset R$ and each $g \in R$ has a unique expression $g = \sum_{0 \leq i,j < p} g_{i,j}^p x^i y^j$ where $g_{i,j} \in R$ for all i and j. Thus we have a ring surjection

$$\begin{aligned} \pi :& R \to R^p \\ g = \sum_{0 \leq i,j < p} & g_{i,j}^p x^i y^j \mapsto g_{0,0}^p \end{aligned}$$

which we see is R-linear.

For $f \in I^{[q]}$.

On the other hand, if $I \subset R$ is an ideal and $f^p \in I^{[p]}$, then there are $a_i, b_i \in R$ so that $a_i \in I$ and $f = \sum a_i b_i^p$. Applying π we see that $\pi(a_i) = c_i^p$ for some i so that

$$f^p = \sum c_i^p b_i^p = \sum (c_i b_i)^p = \left(\sum c_i b_i \right)^p$$

and hence $f = \sum c_i b_i$.

Kunz's Theorem has the following version (or corollary)

Lemma 1.9. *(Kunz) For ideals $I, J \subset R$, we have*

$$(I : J)^{[p]} = I^{[p]} : J^{[p]}$$

since R is regular.

Proof. For those who have seen flatness, this can be seen by the fact that the flat extension $R \to R^{1/p}$ respects colons of ideals (and applying the previous version of Kunz's Theorem). □

2. Hypersurfaces

Now we discuss the role of characteristic p methods in singularity of hypersurfaces. We begin with a very general definition.

Definition 2.1. For a Noetherian local ring (R, m) and a nonzero element $f \in m$, we have that the number

$$\operatorname{mult}(f) := \max\{d \in \mathbb{Z}_{\geq \nVdash} \mid f \in m^d\}$$

is finite (by Krull's Intersection Theorem). $\operatorname{mult}(f)$ is called the *multiplicity of f*. We call f *singular* if and only if $\operatorname{mult} f \geq 2$ (as in f vanishes in m/m^2).

Example 2.2. Let $R = k[[x, y]]$ where k is a field. Then for $0 \neq f \in m$

$$f = f(0) + f_x(0)x + f_y(0)y + g$$

where g is singular. But $f(0) = 0$ as $f \in m$, so f is singular if and only if $0 = f_x(0) = f_y(0)$. △

Objective. We seek a numerical invariant quantifying the severity of a singularity. Now go back to supposing that (R, m) is a regular local ring. Let $0 \neq f \in m$.

$$\text{Naive proposal: } \frac{1}{\text{mult} f}$$

which equals one if and only if f is nonsingular. The worse the singularity f is, the larger $\text{mult} f$ is, and hence the smaller $1/\text{mult} f$ is, and we have $1/\text{mult} f \in (0,1] \cap \mathbb{Q}$.

Exercise 2.3. *Let (R, m) be a regular local ring and let $0 \neq f \in m$. Prove that*

$$\frac{1}{\mathit{mult} f} = \sup \left\{ \frac{N}{t} \middle| N, t \in \mathbb{N}, t \neq 0, f^N \in m^t \right\}.$$

Now continue assuming that (R, m, k) is an F-finite regular local ring of prime characteristic p.

Definition 2.4. Let $0 \neq f \in m$. We define the F-pure threshold of f

$$\text{fpt}(f) := \sup \left\{ \frac{N}{q} \middle| q = p^e, N \geq 0, f^N \notin m^{[q]} \right\}.$$

Exercise 2.5. *Let k be a perfect field and let $R = k[[\underline{x}_n]] := k[[x_1, \ldots, x_n]]$. Let $0 \neq f \in m = (\underline{x}_n) := (x_1, \ldots, x_n)$. Prove that*

$$\frac{1}{\mathit{mult} f} \leq \mathit{fpt} f \leq \frac{n}{\mathit{mult} f}.$$

Exercise 2.6. *Let R be the regular local algebra $R = k[[x, y]]$ over a perfect field k of prime characteristic p. Let $f = y^2 - x^3$. Then*

$$\mathit{fpt} f = \begin{cases} 1/2 & p = 2 \\ 2/3 & p = 3 \\ 5/6 & p \equiv 1 \text{ mod } 6 \\ \frac{5}{6} - \frac{1}{6p} & p \equiv -1 \text{ mod } 6. \end{cases}$$

Example 2.7. (Elliptic curve, more difficult) Let $R = k[[x, y, z]]$ where k is a perfect field of prime characteristic p. An element $\lambda \in k \setminus \{0, 1\}$ defines an elliptic curve in $\mathbb{P}^2_k$. Let

$$f = y^2 z - x(x - z)(x - \lambda z).$$

We have that

$$\text{fpt} f = \begin{cases} 1 & \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i = 0 \\ 1 - \frac{1}{p} & \text{otherwise.} \end{cases}$$

△

Definition 2.8. Let R and f be as per usual ($0 \neq f \in m$ in an F-finite regular local ring (R, m) of prime characteristic p). We can try to study $\text{fpt} f$ by looking at the supremum over one value of q at a time. In this direction, we define

$$v_f(q) := \max\{N \mid f^N \notin m^{[q]}\}$$

and write $v(q) := v_f(q)$ when f is understood.

$v_f(q)$ is called the v-invariant(/s) in the literature.

Remark 2.9. (Basic facts) Fix the notation of the preceding definition. We see the following.

(1) Immediately
$$\mathrm{fpt} f = \sup \left\{ \frac{v_f(q)}{q} \middle| q = p^e \right\}.$$

(2) $f^{v(p)} \notin m^{[p]}$ so that
$$f^{v(p)p} \notin (m^{[p]})^{[p]} = m^{[p^2]}$$
and thus, $v(p)p \leq v(p^2)$, so that
$$\frac{v(p)}{p} \leq \frac{v(p^2)}{p^2}.$$

(3) The sequence $\{v(q)/q \mid q = p^e\}$ is increasing and bounded (as $\mathrm{fpt} f$ is the supremum of these sequence elements), and $v(q) < q$ for all q so that $v(q)/q < 1$ for all q, and hence
$$\mathrm{fpt} f = \lim_{q \to \infty} \frac{v(q)}{q} \leq 1.$$

Theorem 2.10. *Let R, f, and p be as per usual. Then for all q we have*
$$v_f(q) = \lceil fpt(f)q \rceil - 1.$$

Remark 2.11. Let R, f, and p be as per usual. We have that the F-pure threshold $\mathrm{fpt} f$ is positive.

3. BASE p EXPANSIONS AND F-PURE THRESHOLD

3.1. **Base p Expansions.** Let p be a positive prime number and let $\lambda \in (0, 1]$. Then there exists a unique nonterminating expansion of λ in base p
$$\lambda = \sum_{i=1}^{\infty} \frac{\lambda_i}{p^i}$$
for some $\lambda_1, \lambda_2, \ldots \in \{0, \ldots, p-1\}$.

For instance
$$\frac{1}{p} = 0 + \frac{p-1}{p^2} + \frac{p-1}{p^3} + \cdots .$$

For $e \geq 1$, define the truncation of the expansion of λ
$$\langle \lambda \rangle_{p^e} := \sum_{i=1}^{e} \frac{\lambda_i}{p^i}$$

The following lemma relates nicely to the preceding theorem.

Lemma 3.1. *Let p be a positive prime number and let $\lambda \in (0, 1]$. Then for all $e \geq 1$*
$$\langle \lambda \rangle_{p^e} = \frac{\lceil \lambda_e \rceil - 1}{p^e}.$$

3.2. **Connection with F-pure Threshold.**

Throughout the rest of this section, let (R, m, κ) be an F-finite regular local ring of prime characteristic p and let $0 \neq f \in m$.

By the *cut-off* exponent, we mean the v-invariant $v(q) := v_f(q)$. Recall that (by Krull's Intersection Theorem)

$$\cap_q m^{[q]} = 0.$$

Thus $f \in m^{[q_0]}$ for some q_0, so that $v(q_0) \geq 1$ and thus $v(q_0)/q_0 \geq 1/q_0$.

The following statement is very nontrivial.

Theorem 3.2. *(Blickle, Mustaţă, Smith). fptf is a rational number.*

Remark 3.3. Let $\lambda \in (0, 1]$ and consider the base p expansion of λ

$$\lambda = \sum_{e \geq 1} \frac{\lambda_e}{p^e}.$$

We have that $\{\lambda_e \mid e \geq 1\}$ is not eventually zero. We write

$$\lambda = .\lambda_1 \lambda_2 \lambda_3 \cdots$$

so that

$$\langle \lambda \rangle_{p^e} = .\lambda_1 \cdots \lambda_e.$$

Theorem 3.4. *(Hernández)*

We have for all $q = p^e$ that

$$\langle \mathit{fptf} \rangle_e = v(q)/q$$

so that $\langle \mathit{fptf} \rangle_e \to \mathit{fptf}$ as $e \to \infty$.

(one equality is automatic as $v(q)/q$ increases to fpt$f$ as $q \to \infty$).

Sketch of proof.

We review the following facts for ideals $I, J \subset R$.

1. For $g \in R$, we have (under mild conditions) that $g^p \in I^{[p]}$ if and only if $g^q \in I^{[q]}$ for $q \geq p$.
2. We have that $(I : J)^{[p]} = (I^{[p]} : J^{[p]})$ by Kunz's Theorem (since R is regular implies that the extension $R \to R^{1/q}$ is flat and so preserves colons).
3. We always have that
$$(I^{[q]})^{[q']} = I^{[qq']}$$
for all q, q'.

Fix $q = p^s$ and fix $e \geq 1$. By definition we have that $f^{v(q)} \notin m^{[q]}$ so that

$$f^{v(q)+1} \in m^{[q]}.$$

Then by (1) above,

$$f^{(v(q)+1)p^e} = f^{v(q)p^e+p^e} \in m^{[qp^e]},$$

as in, $v(qp^e) < v(q)p^e + p^e$, so that

$$\frac{v(qp^e)}{qp^e} < \frac{v(q)}{q} + \frac{1}{q}.$$

Letting $e \to \infty$ we obtain

$$\mathrm{fpt} f \leq \frac{v(q)+1}{q}$$

and hence $q\mathrm{fpt} f \leq v(q) + 1$.

3.3. **Freedom to move.** We have a fixed guy $0 \neq f \in m$. Fix $q = p^s$ and fix $e \geq 1$. By Krull Intersection again, there exists q_0 such that $f \notin m^{[q_0]}$ (which, by the way, is part of the idea behind positivity of $\mathrm{fpt} f$).

Claim (*Freedom to move*). $f^{v(q)q_0+1} \notin m^{[q_0 q]}$.

Proof. Suppose on the contrary that $f^{v(q)q_0+1} \in m^{[q_0 q]}$. As in,

$$f^{v(q)q_0} f \in \left(m^{[q_0]}\right)^{[q]}.$$

Consequently we have

$$\begin{aligned} f &\in ((m^{[q]})^{[q_0]} : f^{v(q)q_0}) \\ &= (m^{[q]} : f^{v(q)})^{[q_0]} \subset m^{[q_0]} \end{aligned}$$

and $m^{[q]} : f^{v(q)} \subset m$ since $f^{v(q)} \notin m^{[q]}$ (also the equality follows from from the version (2) of Kunz's theorem in the preceding subsection), and this is a contradiction (we had $f \notin m^{[q_0]}$ previously). ☐

It is definitional that $f^{v(q)q_0} \notin m^{[v(q)q_0]}$ and the statement of the preceding claim is that we can move slightly from the known value and still avoid the relevant ideal.

4. Cubic surfaces and Extremal Hypersurfaces

We now discuss the paper *Cubic Surfaces of Characteristic Two* of J. Singh, Vraciu, E. Witt, Z. Kadyrsizova, J. Kenkel, J. page, and K. Smith.

Definition 4.1. Let k be an algebraically closed field of characteristic two, let $S = k[x, y, z, w]$ be the polynomial ring, and let $I \subset S$ be an ideal generated by a single cubic form (homogeneous polynomial) f. Let $X = Z_{\mathbb{P}^3_k}(I)$ be the variety defined by this form. So the coordinate ring of this variety is $S(X) = S/f$.

Question 4.2. *When is the coordinate ring $S(X) = S/f$ F-split (or F-pure)?*

Question 4.3. *If $S(X)$ is not F-pure, what can we say about it?*

Fact. There are exactly 20 cubic (monic) monomials in S, so we may identify f with it's list of 20 coefficients of these monomials, under some fixed ordering, and hence we have an identification

$$\{\text{cubic forms in } S\} \longleftrightarrow k^{20}.$$

Now by forgetting the scaling of f by nonzero scalars (elements of $k^\times$) in both sides of this correspondence we get an identification

$$\{\text{cubic forms in } S \text{ up to nonzero scaling}\} \longleftrightarrow \mathbb{P}^{19}_k.$$

Fix such a cubic form $f \in S$. In order to understand when $S(X) = S/f$ is F-pure, we employ Fedder's Criteria).

Theorem 4.4. *(Fedder's Criteria) Let A be a polynomial ring over a field in prime characteristic and let $g \in A$ be a form. Then A/g is F-pure if and only if $f^{p-1} \in m^p$.*

Applying this to our characteristic two situation, S/f is F-pure if and only if $f \in m^{[2]} = (x_1^2, \ldots, x_4^2)$.

Remark 4.5. From the calculation $m^{[2]} = (x_1^2, \ldots, x_4^2)$, clearly $x_1x_2x_3, x_1x_2x_4, x_1x_3x_4$, and $x_2x_3x_4$ are the only cubic monomials not in $m^{[2]}$. Hence (by Fedder) $S(X) = S/f$ is F-pure if and only if one of these four monomials supports f.

Now applying our previous correspondence, we get the following inclusion of correspondences

$$\begin{tikzcd}
\{\text{cubic surfaces } Z(f) \subset \mathbb{P}^3_k\} \arrow[r, leftrightarrow] & \mathbb{P}^{19}_k \\
\{\text{such } f \text{ s.t. } S/f \text{ is } F\text{-pure}\} \arrow[u] \arrow[r, leftrightarrow] & \{\text{strings of coefficients supported by one of those four monomials}\}\} \arrow[u]
\end{tikzcd}$$

Suppose that S/f is not F-pure. $f \in m^{[q]}$ implies, since f is a cubic form, that

$$f = L_1x_1^2 + L_2x_2^2 + L_3x_3^2 + L_4x_4^2$$

for some linear forms $L_1, \ldots, L_4$.

Theorem 4.6. *(Kadyrsizova et. al. (same authors)) There are only finitely many non-F-pure cubic Hypersurfaces up to linear isomorphism.*

In fact, they describe a complete list of representatives.

Remark 4.7. (Witt) All non-F-pure cubic hypersurfaces f satisfy $\operatorname{fpt} f = 1/2$.

This is the worst possible F-pure threshold such a cubic hypersurface can have.

4.1. **Outside of Cubic Surfaces (Same authors).** Let k be an algebraically closed field (of any prime characteristic p now) and let $S = k[x_1, \dots, x_n]$ be the polynomial ring.

Question 4.8. *Regarding a degree d form $f \in S$, what is the worst ($:\equiv$ minimal) F-pure threshold that f can have, in terms of d only?*

Now fix $d \geq 1$ and a degree d form $f \in S$.

Theorem 4.9. *If f is reduced (irreducible), then*

$$fptf \geq \frac{1}{d-1}.$$

The reduced hypothesis is need for the Frobenius analysis aspect in the argument to work (see the next theorem below for more context for this comment). The idea of the proof is to reduce to the two-variable case by intersecting with linear hyperplanes and using Bertini's Theorem.

Theorem 4.10. *Again let $f \in S$ be irreducible, then*

$$fptf = \frac{1}{d-1}$$

if and only if

$$\textit{there exists } e \geq 0 \textit{ with } d-1 = p^e \textit{ and } f \in m^{[p^e]}$$

(where $m := S_+ = (x_1, \dots, x_n)$).

We note that $f \in m^{[p^e]}$ implies by itself that $\operatorname{fpt} f \leq \frac{1}{p^e}$.

Comments on the preceding theorem. Note how $d - 1 = p^e$ and $f \in m^{[p^e]}$ implies that (as now f is a form of degree $d = p^e + 1$)

$$f = \sum_{i=1}^{n} L_i x_i^{p^e}$$

for some linear forms $L_1, \dots, L_n$. The theorem is also obtained when $e = 0$ as we then just a quadratic form.

Definition 4.11. A *Frobenius form* is a form $f \in S$ $(:= k[x_1, \dots, x_n])$ such that

$$f = \sum_{i=1}^{n} L_i x_i^{p^e}$$

for some $e \geq 0$ and linear forms $L_1, \dots, L_n$.

We make some remarks on the preceding definition.

Remark 4.12. Let S and a Frobenius form $f = \sum_{i=1}^{n} L_i x_i^{p^e}$ be as in the preceding definition. We have for $1 \leq i \leq n$ that

$$L_i = \sum_{j=1}^{n} a_{j,i} x_j$$

for some $a_{j,i} \in k$. Let A be the matrix $A = (a_{j,i})_{j,i} \in k^{n \times n}$ and let $q = p^e$ we see that

$$\begin{aligned} f &= \sum L_i x_i^q \\ &= \sum_i (a_{i,1} x_1 + \cdots \ + a_{i,n} x_n) x_i^q \\ &= (x_1^q \dots x_n^q)(x_1 \dots x_n)^T \\ &= (\underline{x}^{[q]})^T A(\underline{x}) \end{aligned}$$

where $(-)^T$ is matrix transposition. For a matrix $M \in \mathrm{GL}_n(k)$ recall that we get an endomorphism $\phi_M \in \mathrm{End}_k(k[x_1, \dots, x_n])$ given by $(x_1 \dots x_n)^T \mapsto M((x_1 \dots x_n)^T)$ which is an isomorphism since M is invertible (we see readily that $(\phi_M)^{-1} = \phi_{(M^{-1})}$). In other words, ϕ_M is a linear change of coordinates. Factoring with respect to the linear change of coordinates a matrix M and recalling our correspondence $f \leftrightarrow A$ (coming from our calculation that $f = (\underline{x}^{[q]})^T A(\underline{x})$), we see that

$$\phi_M(f) = (M^{[p]} \underline{x}^{[p]})^T A(M\underline{x}) = ((\underline{x}^{[p]})^T (M^{[p]})^T) A(M\underline{x})$$

so that we have a correspondence of operators

$$\phi_M \longleftrightarrow (M^{[p]})^T AM.$$

Remark 4.13. Fix all of the notation of the preceding remark. Recall that each invertible matrix $M \in \mathrm{GL}_n(k)$ is a product of elementary matrices. So we can look a the calculation of the preceding remark, but where M is just an elementary matrix. In this case we see that $(M^{[p]})^T$ is just the p^{th} power of the corresponding row operation (to the elementary matrix M) and M itself acts (by multiplication) as an elementary column operation. This combined allows us to view the operator $(M^{[p]})^T AM$ more easily, as it is just a column operation, followed by A, then a p^{th} power of the corresponding row operation.

Related work. There is related work of J. Singh on test ideals of extremal surfaces, and work of Smith and Vraciu on completely understanding the F-pure threshold bounds in more particular situations.

9.2 Tutorial Problems

Here are the associated tutorial problems.

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WEEK 2: FROBENIUS EXERCISES I

As in the lecture, k is a perfect field of characteristic p, and $(R, \mathfrak{m})$ is an F-finite regular local ring, which we often assume to be the localization of a polynomial ring over k at the homogeneous maximal ideal, or a ring of formal power series over k. Throughout, f stands for an element of $\mathfrak{m}$.

Elementary number theory.

(1) Compute the base p expansion of the rational number $5/6$ for all primes p.
Suggestion: Consider the cases $p = 2, p = 3, p \equiv 1 \bmod 6, p \equiv 5 \bmod 6$ separately.

(2) Verify that if $\lambda \in (0, 1]$, then *the* p^e*-th truncation of* λ, or simply *e-th truncation of* λ when p is clear from context, is $(\lceil \lambda p^e \rceil - 1)/p^e$.

(3) Given $k, \ell, n \in \mathbb{N}$ with $k + \ell = n$, we write $\binom{n}{k,\ell} := \frac{n!}{k!\ell!}$. Consider the base p expansions

$$k = k_0 + k_1 p + \cdots + k_e p^e, \ell = \ell_0 + \ell_1 p + \cdots + \ell_e p^e, n = n_0 + n_1 p + \cdots + n_e p^e$$

with at least one of the terminal coefficients k_e, ℓ_e, n_e nonzero. Prove that

$$\binom{n}{k, \ell} \equiv \binom{n_0}{k_0, \ell_0} \cdots \binom{n_e}{k_e, \ell_e} \bmod p$$

where we interpret $\binom{n_t}{k_t,\ell_t} = 0$ if $k_t + \ell_t \neq n_t$. Conclude that $\binom{n}{k,\ell} \not\equiv 0 \bmod p$ if and only if k and ℓ sum to n *without carrying in base p*. This congruence is known as *Lucas' Theorem.*
Hint: Over $\mathbb{Z}/p\mathbb{Z}$, compute $(x + y)^n$ naively, using the multinomial theorem, then again in steps (guided by the expansion of n, and Frobenius). Compare the coefficients of $x^k y^\ell$.

(4) Recall the *multinomial theorem*, and precisely state an analog for multinomial coefficients.

Basic estimates.

(5) Prove that $\frac{1}{\operatorname{mult}(f)} = \sup\{N/d : N, d \in \mathbb{N}, f^N \notin \mathfrak{m}^d\}$. Briefly explain why if b is an arbitrary positive integer, then this is the same as $\sup\{N/b^e : N, e \in \mathbb{N}, f^N \notin \mathfrak{m}^{b^e}\}$.

(6) Prove that if there are n ambient variables, then $\frac{1}{\operatorname{mult}(f)} \le \operatorname{fpt}(f) \le \frac{n}{\operatorname{mult}(f)}$.
Hint: Identify a uniform regular power of $\mathfrak{m}$ contained in a given Frobenius power of $\mathfrak{m}$.

(7) Consider a grading in which the degree of each ambient variable is a positive integer, not necessarily 1, and suppose that f is a homogeneous polynomial with respect to this grading. For instance, we may take $f = y^2 - x^3$ under the grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Derive a natural upper bound for each $\nu_f(q)$, and consequently, for $\operatorname{fpt}(f)$.

An elementary computation. Consider the formula

$$\operatorname{fpt}(y^2 - x^3) = \begin{cases} \frac{1}{2} & p = 2 \\ \frac{2}{3} & p = 3 \\ \frac{5}{6} & p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p} & p \equiv 5 \bmod 6 \end{cases}$$

(8) Rewrite this in terms of the truncations of $5/6$.

(9) Verify that the formula is correct. *Suggestion*: Compute digit by digit, i.e., start with $\nu_f(p)$.

Connections with Frobenius split rings.

(10) Prove that the following conditions are equivalent.
  (a) R/f is F-split.
  (b) $f^{p-1} \notin \mathfrak{m}^{[p]}$.
  (c) $f^{q-1} \notin \mathfrak{m}^{[q]}$ for all $q = p^e$.
  (d) $\operatorname{fpt}(f) = 1$.

Elliptic curves. Suppose $f = y^2z - x(x-z)(x-\lambda z) \in \mathsf{k}[x,y,z]$ with $\lambda \in \mathsf{k}$ and $\lambda \neq 0, 1$. This is the Legendre form for the equation of an elliptic curve E in $\mathbb{P}^2$.

(11) Prove that $\operatorname{fpt}(f) = 1$ if and only if $\sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i}^2 \lambda^i \neq 0$.

(12) During the lecture, we claimed that if the F-pure threshold is not 1, then it must be $1-(1/p)$. Assume this, and show that it implies the following: If $f^{p-1} \in \mathfrak{m}^{[p]}$, then $f^{p-2} \notin \mathfrak{m}^{[p]}$.

Properties of the F-pure threshold, as a number. Set $\lambda = \operatorname{fpt}(f)$.

(13) Prove that the first digit of the non-terminating base p expansion of λ is less than or equal to every other digit of λ.

(14) Fix a positive integer e, and let μ be the rational number obtained by repeating the first e digits appearing in this expansion of λ. Prove that $\lambda \geq \mu$.

Cubic surfaces. This is in anticipation of an upcoming lecture. By a (projective) *cubic surface*, we mean a homogeneous polynomial of degree 3 in 4 variables. Here, we work in characteristic 2.

(15) Describe the cubic surfaces f such that the quotient R/f is F-split.

(16) Identify a few specific cubic surfaces not among those identified above, and compute (or estimate) their F-pure thresholds.

Frobenius Exercises 2

As throughout the lecture $\Bbbk$ stands for an algebraically closed field of characteristic $p > 0$.

1. Continue working on any problems on the first worksheet that interest you!
2. Given an invertible 2×2 matrix E, describe the induced ring map $\phi_E : \Bbbk[x, y] \to \Bbbk[x, y]$
3. If f is the Frobenius form corresponding to $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is,
$$f = (ax + by)x^2 + (cx + dy)y^2,$$
verify that $\phi_E(f)$ corresponds to the matrix $(E^{[p]})^{tr}AE$.
4. Find an invertible 2×2 matrix M such that
$$\phi_M(x^2y + xy^2) = x^3 + y^3.$$
Hint: Translate this to the matrices associated to these Frobenius forms.
5. Describe all Frobenius forms $f \in \Bbbk[x_1, \cdots, x_n]$ whose associated matrix A has rank 1. How many are there up to linear isomorphism?

   Recall that f and g are equivalent up to linear isomorphism if $\phi_M(f) = g$ for some invertible matrix M.