An Algebraic Condition for a Complex to be Virtual

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Contents

Abstract

The minimal free resolutions of sub-varieties of a toric variety are often larger than is necessary for their geometric applications. Virtual resolutions were defined by Berkesch, Erman, and Smith in the context of toric varieties in [2], and are often given by shorter and thinner complexes than minimal free resolutions. Rather than looking at an exact sequence of free modules resolving an ideal, one considers a complex of free modules which when sheaffified gives a resolution of the associated sheaves.

This project aims to give an introduction to resolutions, both minimal and virtual, and to give an algebraic condition on a complex to guarantee it is a virtual resolution.

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CHAPTER 1

Introduction

A free resolution of an $R-$module M is an exact sequence of free $R-$modules whose $0-$th homology is isomorphic to M. Resolutions naturally arises in algebraic geometry because there is a correspondence between closed sub-varieties of projective space $\mathbb{P}^n$ over an algebraically closed field $\mathbb{K}$, and proper radical homogeneous ideals in $R = \mathbb{K}[x_0, \ldots, x_n]$. To every radical homogeneous ideal there is an associated minimal free resolution and the minimal free resolution gives geometric information about the associated variety (for instance its dimension). Chapter 2 focuses on the algebraic background of minimal free resolutions used in the studying of sub-varieties of $\mathbb{P}^n$.

A natural next step would be to consider products of projective spaces. In order to study sub-varieties of a product of projective spaces algebraically, we first need to find the appropriate polynomial ring whose radical ideals correspond to the sub-varieties. A product of projective spaces is an example of a toric variety. In Chapter 3 we closely follow [5] to define a toric variety and show how to construct one from a fan of cones. Given a cone σ, one considers its dual cone $\hat{\sigma}$ which gives an affine semi-group S_σ, which in turn gives a toric variety $U_\sigma = \text{Spec}(\mathbb{K}[S_\sigma])$.

The structure of the fan Σ gives the necessary information to glue together the toric varieties U_σ into an abstract toric variety which we denote X_Σ. The fan also gives the necessary information to construct the total coordinate ring S_Σ. Furthermore there is a correspondence between certain radical homogeneous ideals of S_Σ and the sub-varieties of X_Σ. In general, S_Σ is graded by the class group $Cl(X_\Sigma)$. In the case of a product of projective spaces, the class group is $\mathbb{Z}^r$ where r is the number of projective spaces appearing in the product.

We can then use the algebraic techniques discussed in Chapter 2 to study the homogeneous radical ideals of S_Σ (adapting the definitions of a minimal free resolution to the multi-graded case), however this usually leads to complexes which are larger than is necessary. In Chapter 4 we give the definition of a virtual resolution, first introduced by Berkesch et al. in [2]. This is a new definition and not very much is known about virtual resolutions. However, in Section 5 of [2], several geometric applications of virtual resolutions are given. Instead of considering an exact sequence of free modules, we consider a complex which when sheaffified gives an exact sequence of sheaves.

Example 1.4 in [2] demonstrates how minimal free resolutions are often much larger than virtual resolutions. A hyperelliptic curve curve of genus 4 can be embedded into $\mathbb{P}^1 \times \mathbb{P}^2$. Letting I be the corresponding ideal, S/I has minimal free resolution

$$S \leftarrow \begin{matrix} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8) \end{matrix} \leftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7) \\ \oplus \\ S(-1,-8)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8) \end{matrix} \leftarrow S(-3,-7) \leftarrow 0.$$

However S/I has a virtual resolution

$$S \leftarrow \begin{matrix} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-2,-3)^2 \end{matrix} \leftarrow S(-3,-3)^3 \leftarrow 0,$$

which is both shorter and thinner than the minimal free resolution.

Every minimal free resolution is a virtual resolution. However the sequence of sheaves being exact is less strict of a condition than the sequence of free modules being exact. In Section 2 of Chapter 4 we show that the sequence of sheaves being exact corresponds to the sequence of modules having limited homology. Our main results are: Theorem 4.9, which gives an algebraic condition for a complex F to be a virtual resolution of M, and Theorem 4.10 which specializes Theorem 4.9 to the case when the first free module in the complex is S and we are resolving S/I.

The reason for considering the special case of Theorem 4.10 is that the two methods, which are given in [2], of producing virtual resolutions come from minimal free resolutions. The first method constructs a virtual resolution of an $S-$module M by removing summands from the minimal free resolution. In general, this method shortens and thins out the complex, but does not leave it exact. The second method applies to sets of points, and guarantees that there is an ideal Q such that the minimal free resolution of $S/(I \cap Q)$ is a virtual resolution of S/I with length equal to the dimension of $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

Finally, we consider examples of virtual resolutions in $\mathbb{P}^1 \times \mathbb{P}^1$.

CHAPTER 2

Background

In this chapter we begin by reviewing projective varieties and their correspondence with radical homogeneous ideals. We then follow [4] to discuss algebraic techniques which are used to study these ideals.

1. Projective Varieties

Let $\mathbb{K}$ be an algebraically closed field, and let $\mathbb{P}^n$ be the $\mathbb{K}-$projective space with homogeneous coordinates $[x_0 : \ldots : x_n]$. Projective space can be thought of as the quotient

$$\mathbb{P}^n = (\mathbb{K}^{n+1} \smallsetminus 0)/\sim$$

where two vectors are identified if they are non-zero multiples of each other. Consider a subset $V \subset \mathbb{P}^n$, and map into the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_n]$ by looking at the homogeneous polynomials which vanish on V. We define the homogeneous ideal $\mathbb{I}(V)$ to be the ideal generated by the homogeneous polynomials vanishing on V, that is,

$$\mathbb{I}(V) = \langle \{f \in R : f \text{ is homogeneous and } f(x) = 0 \text{ for all } x \in V\} \rangle .$$

Similarly, if $I \subset R$ is a homogeneous ideal, we can define $\mathbb{V}(I) \subset \mathbb{P}^n$ to be the set

$$\mathbb{V}(I) = \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\},$$

where T is the set of homogeneous elements in I. This gives us a bijective correspondence between sub-varieties $V \subset \mathbb{P}^n$ and radical homogeneous ideals $I \subset R$. Recall that an ideal is homogeneous if it is generated by homogeneous elements (that is polynomials where every monomial has the same total degree), and an ideal is radical if $I = \sqrt{I}$, where $\sqrt{I} = \{f \in R : \text{ there is some } n \in \mathbb{N} \text{ such that } f^n \in I\}$.

2.1. Theorem. *(see exercises 2.1-2.4 in [8]).*

Let $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle \subset R = k[x_0, \ldots, x_n]$. There is a bijective correspondence

$$\{ \text{ non-empty and closed sub-varieties of } \mathbb{P}^n\} \iff \{\text{homogeneous radical ideals not equal to } \mathfrak{m}\}$$

The ideal $\mathfrak{m}$ is called the irrelevant ideal of $\mathbb{P}^n$.

The *saturation* of an ideal by $\mathfrak{m}$ is the ideal

$$(I : \mathfrak{m}^\infty) = \{f \in R : f\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}.$$

Note that every ideal in the correspondence of Theorem 2.1 is equal to its saturation.

2.2. Lemma. *If $I \subsetneq R$ is a radical homogeneous ideal not equal to $\mathfrak{m}$, then we have*

$$I = (I : \mathfrak{m}^\infty).$$

Proof. Since I is radical, it is the intersection of the minimal prime ideals containing it (see Proposition 1.14 in [1]). Every such prime ideal P is contained in $\mathfrak{m}$ because P also is homogeneous. Indeed, since I is homogeneous we have $I \subset \hat{P}$ where $\hat{P}$ is the ideal generated by the homogeneous elements in P. Since it suffices to check the primality of a homogeneous ideal on homogeneous elements of R, we have that $\hat{P}$ is prime. Indeed if $ab \in \hat{P}$ with a and b homogeneous, then $ab \in P$ so either a or b is in P. Since the element is homogeneous it is also in $\hat{P}$. Since P is a minimal prime we conclude that $P = \hat{P}$, so it is homogeneous and hence must be contained in $\mathfrak{m}$. Since $I \neq \mathfrak{m}$, there is a prime appearing in the decomposition of I which is strictly contained in $\mathfrak{m}$, so $\mathfrak{m}$ is not a minimal prime containing I.

Its clear that $I \subset (I : \mathfrak{m}^\infty)$. Moreover if we have

$$f \in (I : \mathfrak{m}^\infty),$$

then there exists $n \in \mathbb{N}$ such that for each x_i we have

$$f x_i^n \in I.$$

Now, if $f \notin I$ then $f \notin P$ for some minimal prime ideal P containing I. Then $x_i^n \in P$ and hence $x_i \in P$ for each i. This shows that $\mathfrak{m} = P$ which is a contradiction. We conclude that $f \in I$ and that $I = (I : \mathfrak{m}^\infty)$.

□

This correspondence shows us that we can study sub-varieties $V \subset \mathbb{P}^n$ by studying the $\mathfrak{m}-$saturated radical homogeneous ideals of R. In sections 2 and 3 we outline some of the tools used in studying these ideals.

2. Free Resolutions

Let M be a finitely generated $R-$module. If we wish to study the structure of M, we start by choosing a set of generators, $f_1, \ldots, f_m$, for M. This is equivalent to choosing a surjective $R-$module homomorphism

$$R^m \xrightarrow{\varphi_0} M$$

$$e_i \mapsto f_i,$$

where e_i is the $i-$th standard basis vector in the free module R^m,

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

By the First Isomorphism Theorem (Theorem 4 of section 10.2 in [6]), we then have

$$M \cong R^m / \ker(\varphi_0).$$

If $\sum_{i=1}^{m} a_i e_i \in \ker(\varphi_0)$, then we have

$$0 = \varphi_0 \left(\sum_{i=1}^{m} a_i e_i \right) = \sum_{i=1}^{m} a_i \varphi_0(e_i) = \sum_{i=1}^{m} a_i f_i.$$

The elements

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \ker(\varphi_0)$$

are called *syzygies* of $f_1, \ldots, f_m$, and the kernel, $\ker(\varphi_0)$, is called the (first) *syzygy module* of M.

We see that to understand the structure of M we need to understand both the generators, $f_1, \ldots, f_m$, of M and also the relations between them.

Since these relations form the kernel of the $R-$module homomorphism, $R^m \xrightarrow{\varphi_0} M$, they form a finitely generated submodule of R^m (since R is Noetherian). To understand this submodule, we must then repeat the process of choosing generators and considering the relations between them. Again, choosing a set of generators is equivalent to choosing a surjective map,

$$R^n \xrightarrow{\varphi_1} \ker(\varphi_0),$$

and the relations between the generators form the kernel of this map.

This gives an exact sequence

$$R^n \xrightarrow{\varphi_1} R^m \xrightarrow{\varphi_0} M \to 0.$$

Continuing in this way gives rise to the definition of a resolution.

2.3. Definition. (Definition 1.9 in Chapter 6 of [4]) A free resolution of an $R-$module M is an exact sequence

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

where each F_i is a free $R-$module.

If the resolution is eventually 0, that is, if there is some l such that $F_i = 0$ for all $i > l$ and $F_l \neq 0$, then the resolution is said to be *finite of length l*.

When the module M is finitely generated and each F_i is isomorphic to R^t for some t, we can represent the maps φ_l by matrices over R (in the exact same way that we represent linear transformations of finite dimensional vector spaces).

Of course this definition can be applied to any ring R and any $R-$module M. A natural question to ask is when an $R-$module has a finite free resolution. The following example shows that this need not be the case when R is a quotient ring, even though the module in question is finitely generated.

2.4. Example. (Exercise 11 in Section 6.1 of [4])

Consider $R = \mathbb{K}[x]/\langle x^2\rangle$ and $M = \langle x\rangle \subset R$. The kernel of the map $R \to M$ given by multiplication by x,

$$\varphi\colon R \to M, \quad f \mapsto xf$$

is the module M itself. Indeed,

$$R = \{a\overline{x} + b : a, b \in \mathbb{K}\},$$

and if

$$\overline{x}(a\overline{x} + b) = 0,$$

then

$$ax^2 + bx \in \langle x^2\rangle,$$

so $b = 0$.

This gives an infinite resolution

$$\cdots \to R \to R \to \cdots \to R \to M \to 0,$$

where each map is multiplication by x.

In some sense, every resolution of M contains this one. Suppose that we have any resolution of M,

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0.$$

Since φ_0 is surjective, there must exist some element $f_0 \neq 0$ mapping to $\overline{x} \in M$. Hence $\varphi_0(\overline{x}f_0) = \overline{x^2} = 0$, so $\overline{x}f_0 \in \ker(\varphi_0)$. Thinking of the free module F_0 as R^t for some t, if $\overline{x}f_0 = 0$, then we must have each component of xf_0 in $\langle x^2\rangle$, and hence we could factor f_0 as $f_0 = \overline{x}\tilde{f}_0$. However, we would then have $\varphi_0(\overline{x}\tilde{f}_0) = \overline{x}\varphi_0(\tilde{f}_0)$, and since $\varphi_0(\tilde{f}_0) \in \langle\overline{x}\rangle$, we would have $\varphi_0(f_0) = 0$ which is a contradiction. This shows that the kernel of φ_0 is non-zero.

Now since $\text{im}(\varphi_1) = \ker(\varphi_0)$, there must be some $f_1 \in F_1$ mapping to $\overline{x}f_0$. We again have that $\overline{x}f_1 \neq 0$ showing that the kernel, $\ker(\varphi_1)$ is non-trivial. Indeed, suppose that $\overline{x}f_1 = 0$. Then we could again factor f_1 as $f_1 = \overline{x}\tilde{f}_1$ (thinking of F_1 as R^s for some s). Then in each component, $i = 1, \ldots, t$, we have

$$\overline{x}\varphi_1(\tilde{f}_1)_i = \overline{x}(f_0)_i,$$

in R. This shows that in $\mathbb{K}[x]$ we have

$$x\varphi_1(\tilde{f}_1)_i = x(f_0)_i + g_i x^2,$$

for some $g_i \in \mathbb{K}[x]$, and hence

$$\varphi_1(\tilde{f}_1) - f_0 = \overline{x}g$$

where g is the vector consisting of the classes of the g_i. Again since $\varphi_0(g) \in \langle \overline{x} \rangle$, we have

$$\varphi_1(\tilde{f}_1) - f_0 \in \ker(\varphi_0) = \operatorname{im}(\varphi_1).$$

So we can find some $g_1 \in F_1$ such that $\varphi_1(g_1) = \varphi_1(\tilde{f}_1) - f_0$, showing that $f_0 \in \operatorname{im}(\varphi_1) = \ker(\varphi_0)$, and this is a contradiction since $\varphi_0(f_0) = \overline{x}$.

Using an inductive argument, we can show that the kernel of each map is non-trivial implying that the resolution must be infinite. The inductive argument is as follows:

The claim is that for each element f_l mapping to xf_{l-1} we have $xf_l \neq 0$ showing that the kernel is non-trivial. If $xf_l = 0$ then we could factor $f_l = x\tilde{f}_l$ and conclude that

$$\varphi_l(\tilde{f}_l) - f_{l-1} = xg_l$$

for some g_l. However, f_{l-1} maps to xf_{l-2}, and $\varphi_{l-1} \circ \varphi_l = 0$ so we get $\varphi_{l-1}(-xg) = xf_{l-2}$ and this contradicts the inductive assumption since $x(-xg) = 0$.

$\Diamond$

Fortunately, the context we are interested in is when R is a polynomial ring over a field, that is $R = \mathbb{K}[x_0, \ldots, x_n]$, and in this case we have the following result:

2.5. Theorem. *(Theorem 2.1 in Chapter 6 of [4]) Let $R = \mathbb{K}[x_0, \ldots, x_n]$ and M a finitely generated $R-$module. Then M has a free resolution of length less than or equal to $n + 1$.*

The proof of Theorem 2.5 in [4] gives an algorithm to compute free resolutions, but it very quickly becomes unrealistic to compute by hand. *Macaulay2* [7] implements many algorithms useful for computations in algebraic geometry. The following example uses *Macaulay2* and demonstrates how a resolution depends on the generators chosen at each stage.

2.6. Example. (Exercise 1 of Section 3 in [4])

Let $R = \mathbb{K}[x, y]$ and

$$I = \langle x^2 - x, xy, y^2 - y \rangle \subset R.$$

Using *Macaulay2*, we compute the generators of the first syzygy module and get the partial resolution,

$$R^3 \xrightarrow{\begin{pmatrix} y & 0 & y^2-y \\ 1-x & y-1 & 0 \\ 0 & -x & x-x^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2-x & xy & y^2-y \end{pmatrix}} I \to 0.$$

If we label the columns of the matrix g_1, g_2, g_3, then we see that

$$(y-1)g_1 + (x-1)g_2 = g_3.$$

Since g_1 and g_2 are independent, we conclude that

$$\begin{pmatrix} y-1 \\ x-1 \\ -1 \end{pmatrix}$$

is a generator for the second syzygy module, and we get a resolution

$$0 \to R \xrightarrow{\begin{pmatrix} y-1 \\ x-1 \\ -1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & 0 & y^2-y \\ 1-x & y-1 & 0 \\ 0 & -x & x-x^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2-x & xy & y^2-y \end{pmatrix}} I \to 0.$$

We see that this resolution has length 2. Alternatively, since g_3 is a combination of g_1 and g_2, we can remove it from the set of generators and we get a shorter resolution,

$$0 \to R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 1-x & y-1 \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2-x & xy & y^2-y \end{pmatrix}} I \to 0.$$

$\Diamond$

3. Graded Resolutions

The ring $R = \mathbb{K}[x_0, \ldots, x_n]$ has structure which we have not yet used, namely it is a graded ring with respect to degree. That is, we have a direct sum decomposition

$$R = \bigoplus_{s \geq 0} R_s,$$

where R_s is the additive subgroup of homogeneous polynomials of degree s along with 0. Note that R_0 is just the field $\mathbb{K}$.

2.7. Definition. (Definition 3.2 in Chapter 6 of [4]) An $R-$module M is said to be graded (over $\mathbb{Z}$) if there is a collection of subgroups, $M_t \subset M$, satisfying

$$M = \bigoplus_{t \in \mathbb{Z}} M_t,$$

and $R_s M_t \subset M_{s+t}$ for all $s \geq 0$ and $t \in \mathbb{Z}$.

A homomorphism of graded $R-$modules $\varphi\colon M \to N$ is said to be *graded of degree* d if $\varphi(M_t) \subset N_{t+d}$ for all t.

If a module M is graded, then we can form a "twisted" module $M(d)$ by taking

$$M(d)_t = M_{t+d}.$$

We can also form twisted free modules by taking a direct sum

$$R(d_1) \oplus R(d_2) \oplus \cdots \oplus R(d_m),$$

where

$$[R(d_1) \oplus R(d_2) \oplus \cdots \oplus R(d_m)]_t = R(d_1)_t \oplus R(d_2)_t \oplus \cdots \oplus R(d_m)_t.$$

This leads to the definition of a graded resolution.

2.8. Definition. (Definition 3.7 in Chapter 6 of [4]) A *graded free resolution* of a graded $R-$module M is an exact sequence,

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0,$$

such that each F_i is a graded free $R-$module and each φ_i is a graded homomorphism of degree 0.

The simplest source of graded modules are ideals of R generated by homogeneous elements. We can "homogenize" the ideal from Example 2.6 by adding a new variable z. This gives us the following:

2.9. Example. (Exercise 4 from Section 3 of Chapter 6 in [4]) Let $S = \mathbb{K}[x, y, z]$ and $I = \langle x^2 - xz, xy, y^2 - yz\rangle \subset S$. Again using *Macaulay2* we compute the generators of the first syzygy module to be

$$g_1 = \begin{pmatrix} y \\ z - x \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ y - z \\ -x \end{pmatrix}, \quad g_3 = \begin{pmatrix} y^2 - yz \\ 0 \\ xz - x^2 \end{pmatrix}.$$

We see that $g_3 = (y - z)g_1 + (x - z)g_2$, and g_1 and g_2 are independent. This shows that

$$\begin{pmatrix} y - z \\ x - z \\ -1 \end{pmatrix}$$

generates the syzygies of the first syzygy module. This gives us a graded resolution,

$$0 \to R(-4) \xrightarrow{\begin{pmatrix} y-z \\ x-z \\ -1 \end{pmatrix}} \begin{matrix} R(-3)^2 \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 & y^2-yz \\ z-x & y-z & 0 \\ 0 & -x & xz-x^2 \end{pmatrix}} R(-2)^3$$

$$\xrightarrow{(x^2-xz,\, xy,\, y^2-yz)} I \to 0.$$

The twists are computed in ascending order starting at $F_0 = R^3$. We see that every generator is homogeneous of degree 2, so the homomorphism mapping onto I raises degree by 2. We compensate by shifting the degree of R^3 by -2. At the next stage, the first two columns of of the matrix are homogeneous of degree 1, so we need to shift two copies of R in F_1 by -1. Since F_0 has already been shifted by -2, we get a total shift of -3. Similarly, the last column of the second matrix has degree 2, so we must shift by -4. We continue in this way until every map in the resolution has degree 0.

Since g_3 is a combination of g_1 and g_2, we could leave this column out of the graded resolution and obtain a new graded resolution,

$$0 \to R^2(-3) \xrightarrow{\begin{pmatrix} y & 0 \\ z-x & y-z \\ 0 & -x \end{pmatrix}} R^3(-2) \xrightarrow{(x^2-xz,\, xy,\, y^2-yz)} I \to 0.$$

Note that these two resolutions are exactly the homogenized resolutions from Example 2.6. $\diamondsuit$

2.10. Definition. (Proposition 3.10 in Chapter 6 of [4]) Let

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$$

be a graded free resolution of an $R-$module M. The resolution is said to be *minimal* if each map φ_i maps the standard basis of F_i onto a minimal generating set of $\operatorname{im}(\varphi_i)$.

Alternatively, we could use an equivalent definition;

2.11. Theorem. *(Definition* 3.9 *in Chapter 6 of [4])*

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$$

is minimal if and only if for every map φ_i with $i \geq 1$, the matrix representing φ_i does not contain any nonzero constant entries.

2.12. Remark. We can state the definition of a minimal free resolution more compactly as follows:

A minimal free resolution of a graded $R-$module M is a complex of graded free $R-$modules

$$F : F_0 \xleftarrow{\varphi_1} F_1 \leftarrow \cdots \xleftarrow{\varphi_l} F_l \cdots$$

with $H_0(F) \cong M$ and $H_i(F) = 0$ for all $i \geq 1$, and each φ_l is a degree 0 map satisfying $\mathrm{im}(\varphi_l) \subset \mathfrak{m}F_{l-1}$.

We also have the syzygy theorem for graded resolutions. The proof of Theorem 2.5 given in [4] easily adapts to the graded case.

2.13. Theorem. *(Theorem 3.8 in Chapter 6 of [4]) Let M be a finitely generated graded $\mathbb{K}[x_0, \ldots, x_n]$ module. Then there exists a graded resolution of M of length less than or equal to $n+1$.*

The advantage of a minimal resolution is that it is unique up to isomorphism in the following sense.

2.14. Definition. (Definition 3.11 in Chapter 6 of [4]) Let

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$$

and

$$\cdots \to G_l \xrightarrow{\psi_l} G_{l-1} \to \cdots \to G_0 \xrightarrow{\psi_0} M \to 0$$

be two graded free resolutions of M. These resolutions are said to be *isomorphic* if there exists degree 0 isomorphisms $\alpha_i \colon F_i \to G_i$ for each $i \geq 0$ which give rise to a commutative ladder diagram,

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_l & \xrightarrow{\varphi_l} & F_{l-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \xrightarrow{\varphi_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_l} & & \downarrow{\scriptstyle \alpha_{l-1}} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\
\cdots & \longrightarrow & G_l & \xrightarrow{\psi_l} & G_{l-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \xrightarrow{\psi_0} & M & \longrightarrow & 0
\end{array}$$

2.15. Theorem. *(Theorem 3.13 in Chapter 6 of [4]) Let M be a finitely generated graded $R-$module. Any two minimal free resolutions of M are isomorphic.*

Even though we did not use the property of the resolution being graded in definition 2.10, the idea of a minimal free resolution of a non-graded module is not useful. This is because theorem 2.15 only holds if the minimal resolutions are graded. To illustrate this, consider the following example.

2.16. Example. Let $R = \mathbb{K}[x]$ and $M = \langle x \rangle$. We can choose two different minimal generating sets for M and end up with non-isomorphic "minimal" resolutions. Of course, one of them is not actually minimal because it is not graded.

Consider the sets of minimal generators $\{x^2 + x, x^2\}$ and $\{x\}$. We construct two resolutions.

The syzygy module of $\langle x^2 + x, x^2 \rangle$ is generated by

$$\begin{pmatrix} x \\ -x-1 \end{pmatrix},$$

so we get a resolution

$$0 \to R \xrightarrow{\begin{pmatrix} x \\ -x-1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2+x & x^2 \end{pmatrix}} M \to 0.$$

However from the generator x we also get a resolution,

$$0 \to R \xrightarrow{x} M \to 0.$$

The two resolutions are not even the same size so they cannot be isomorphic, yet each map takes the standard basis onto a minimal generating set of its image. The first resolution contains non-homogeneous elements in the matrices, so it cannot be a graded resolution, whereas the second resolution is a minimal resolution if we take the shifted grading,

$$0 \to R(-1) \xrightarrow{x} M \to 0.$$

$\Diamond$

CHAPTER 3

Toric Varieties

1. The Definition of a Toric Variety.

Before defining toric varieties, we revisit projective varieties as an example. Recall the notation from Chapter 2. The $\mathbb{K}-$projective space with homogeneous coordinates $[x_0 : \ldots : x_n]$ is denoted $\mathbb{P}^n$, R is the polynomial ring $\mathbb{K}[x_0, \cdots, x_n]$, a variety $V \subset \mathbb{P}^n$ corresponds to an ideal

$$\mathbb{I}(V) = \{f \in R : \ f \text{ is homogeneous and } f(x) = 0 \text{ for all } x \in V\},$$

and a homogeneous ideal $I \subset R$ corresponds to the variety

$$\mathbb{V}(I) = \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\},$$

where $T \subset I$ is the set of homogeneous elements in I.

If $V \subset \mathbb{P}^n$ is a sub-variety, then we can represent V as the union of affine Zaraski open sets (for more details see chapter 2 and 3 of [5]). Indeed, let $U_i = \mathbb{P}^n \smallsetminus \mathbb{V}(x_i)$. The map

$$[a_0 : \cdots : a_n] \longmapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

gives an isomorphism from U_i to $\mathbb{K}^n$. Then $V \cap U_i$ is a Zariski open subset of V. The map above takes $V \cap U_i$ to the sub-variety of $\mathbb{K}^n$ defined by

$$f\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) = 0$$

as f ranges over all homogeneous polynomials in $\mathbb{I}(V)$. Since the U_i cover $\mathbb{P}^n$, these affine pieces $V \cap U_i$ cover all of V, and hence V decomposes as a union of these affine pieces.

We now consider these sets U_i more carefully. Each U_i is a copy of $\mathbb{K}^n$ in a different set of variables (the x_j/x_i). For each $i \neq j$, $i, j \in \{0, \ldots, n\}$, we have the open subsets

$$(U_i)_{\frac{x_j}{x_i}} \subset U_i, \quad (U_j)_{\frac{x_i}{x_j}} \subset U_j,$$

where $(U_i)_{\frac{x_j}{x_i}}$ is the set

$$(U_i)_{\frac{x_j}{x_i}} = \left\{p \in U_i : \frac{x_j}{x_i}(p) \neq 0\right\}.$$

This Zariski open set is itself an affine variety (see section 1.0 of [5]). The coordinate rings of these affine varieties are simply given by the localization of the coordinate rings of each U_i (see Lemma 2.0.3 in [5]),

$$\mathbb{K}\left[\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}}.$$

The elements of this localized ring are ratios of polynomials over a power of $\frac{x_j}{x_i}$,

$$\frac{f}{(x_j/x_i)^t},\quad t\geq 0.$$

We then get a $\mathbb{K}-$algebra isomorphism

$$g_{ji}^*\colon \mathbb{K}\left[\frac{x_0}{x_j},\ldots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\ldots,\frac{x_n}{x_j}\right]_{\frac{x_i}{x_j}} \longrightarrow \mathbb{K}\left[\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}},$$

given by

$$\frac{x_k}{x_j}\mapsto \frac{x_k}{x_i}/\frac{x_j}{x_i},$$

for $k\neq j$ and

$$\left(\frac{x_i}{x_j}\right)^{-1}\mapsto \frac{x_j}{x_i}.$$

Hence we get an isomorphism of affine varieties (see Corollary 3.7 of [8])

$$g_{ji}\colon (U_i)_{\frac{x_j}{x_i}}\to (U_j)_{\frac{x_i}{x_j}}.$$

These affine varieties $(U_i)_{\frac{x_j}{x_i}}$ and $(U_j)_{\frac{x_i}{x_j}}$ are the same open set $U_i\cap U_j\subset\mathbb{P}^n$ since they are both the set of points $p\in\mathbb{P}^n$ which have both coordinates x_i and x_j non-zero. The point in viewing it this way, is that we can start with the affine varieties U_i and glue them together over their intersection using the isomorphisms g_{ij}. This construction works since $g_{ij}=g_{ji}^{-1}$ and $g_{ki}=g_{kj}\circ g_{ji}$.

For projective space, we also have the quotient construction,

$$\mathbb{P}^n=(\mathbb{K}^{n+1}\smallsetminus 0)/\mathbb{K}^*$$

where $\mathbb{K}^*$ acts on $\mathbb{K}^{n+1}$ by scalar multiplication. Note that the set we remove from the affine space $\mathbb{K}^{n+1}$ is the vanishing set of the irrelevant ideal $\mathfrak{m}=\langle x_0,\ldots,x_n\rangle\subset\mathbb{K}[x_0,\ldots,x_n]$. This gives us two ways to think about projective space; as the quotient of an affine space (minus a certain set) by a group action, or as the gluing together of finitely many affine varieties.

More generally, suppose that we have finitely many affine varieties V_α, and for all pairs α,β, we have Zariski open subsets $V_{\beta\alpha}\subset V_\alpha$ and isomorphisms $g_{\beta\alpha}\colon V_{\beta\alpha}\to V_{\alpha\beta}$ which satisfy the compatibility conditions:

(1) $g_{\alpha\beta}=g_{\beta\alpha}^{-1}$,

(2) $g_{\beta\alpha}(V_{\beta\alpha}\cap V_{\gamma\alpha})=V_{\alpha\beta}\cap V_{\gamma\beta}$ and $g_{\gamma\alpha}=g_{\gamma\beta}\circ g_{\beta\alpha}$ on $V_{\beta\alpha}\cap V_{\gamma\alpha}$.

Then let Y be the disjoint union of the V_α and $\sim$ a relation on Y where $a \sim b$ if and only if there is some α and β such that $a \in V_\alpha$, $b \in V_\beta$, and $g_{\beta\alpha}(a) = b$. The first condition shows that $\sim$ is reflexive and symmetric and the second shows that it is transitive. Hence we can form the quotient space $X = Y/\sim$ with the quotient topology. This quotient space X locally looks like an affine variety since for each α we can take

$$U_\alpha = \{\overline{a} \in X : a \in V_\alpha\},$$

and the quotient map

$$h_\alpha \colon V_\alpha \to U_\alpha, \quad h(a) = \overline{a}$$

is a homeomorphism.

3.1. Definition. (Definition 3.0.5 from [5]) The space $X = Y/\sim$ from above is called an abstract variety.

A Zariski closed subset of such a variety is a *sub-variety*. A variety is said to be *irreducible* if it is not the union of two proper sub-varieties.

3.2. Example. A sub-variety $V \subset \mathbb{P}^n$ is also an abstract variety by taking the Zariski open sets $V_i = V \cap U_i$, and gluing them together similar to above.

$\Diamond$

3.3. Definition. (Definition 3.1.1 in [5]) A *toric variety* is an irreducible variety X which contains a torus $T = (\mathbb{K}^*)^m$ as a Zariski open subset such that the action of T on itself extends to an action of T on X which is algebraic (that is every element $t \in T$ gives a homomorphism from X to X).

3.4. Example. $\mathbb{P}^n$ is a toric variety. Indeed, it has the torus

$$\begin{aligned} T_{\mathbb{P}^n} = \mathbb{P}^n \setminus \mathbb{V}(x_0 \cdots x_n) &= \{[a_0 : \cdots : a_n] \in \mathbb{P}^n : a_0 \cdots a_n \neq 0\} \\ &= \{[1 : t_1 : \cdots : t_n] \in \mathbb{P}^n : t_1, \dots t_n \in \mathbb{K}^*\} \\ &\cong (\mathbb{K}^*)^n \end{aligned}$$

as a Zariski open subset. Then we extend the action of $T_{\mathbb{P}^n}$ by

$$[1 : t_1 : \cdots : t_n] \cdot [a_0 : \cdots : a_n] = [a_0 : t_1 a_1 : \cdots : t_n a_n].$$

A projective variety is irreducible if and only if its associated ideal is prime (see Exercise 2.4 in [8]). The only homogeneous polynomial which vanishes on all of $\mathbb{P}^n$ is the zero polynomial, showing that $\mathbb{I}(\mathbb{P}^n) = 0$. Thus, $\mathbb{P}^n$ is irreducible and hence a toric variety.

$\Diamond$

2. The Affine Toric Variety of an Affine Semi-group

Let $T = (\mathbb{K}^*)^n$ be a torus and consider its characters (group homomorphisms) $\chi \colon T \to \mathbb{K}^*$. It turns out (see section 16 of [9]) that every character of $(\mathbb{K}^*)^n$ arises as one of the form

$$\chi^m(t_1, \dots, t_n) = t_1^{a_1} \cdots t_n^{a_n}, \quad m = (a_1, \dots, a_n) \in \mathbb{Z}^n,$$

and hence the characters of T form a group isomorphic to $\mathbb{Z}^n$. This group is the *character lattice* of T.

Given a torus T with character lattice M and a finite subset $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$, we define a map

$$\Phi_{\mathcal{A}} \colon T \to \mathbb{K}^s$$

by

$$\Phi_{\mathcal{A}}(t) = (\chi^{m_1}(t), \dots, \chi^{m_s}(t)).$$

Let $Y_{\mathcal{A}}$ be the Zariski closure of this map. The $Y_{\mathcal{A}}$ is an affine toric variety by Proposition 1.1.8 in [5].

3.5. Definition. (See section 1.1 of [5]) An *affine semi-group* is a semi-group (a set with an associative binary operation and an identity) with the following further requirements:

(1) The binary operation is commutative.

(2) The semi-group is finitely generated.

(3) The semi-group can be embedded into a lattice M.

An example of an affine semi-group is $\mathbb{N}^n \subset \mathbb{Z}^n$. Given any lattice M and any finite set $\mathcal{A} \subset M$, we have the semi-group $\mathbb{N}\mathcal{A} = \{\sum_{m \in \mathcal{A}} a_m m : a_m \in \mathbb{N}\} \subset M$. Up to isomorphism, all semi-groups are of this form (since by definition they must be able to be embedded into a lattice).

From an affine semi-group S, we can construct the semi-group algebra $\mathbb{K}[S]$ as follows.

$$\mathbb{K}[S] = \left\{ \sum_{m \in S} c_m \chi^m : c_m \in \mathbb{K} \text{ and } c_m = 0 \text{ for all but finitely many } m \right\}$$

with multiplication induced by the semi-group structure, $\chi^m \chi^{m'} = \chi^{m+m'}$. If $S = \mathbb{N}\mathcal{A}$, where $\mathcal{A} = \{m_1, \dots, m_s\}$, then we have

$$\mathbb{K}[S] = \mathbb{K}[\chi^{m_1}, \dots, \chi^{m_s}]. \tag{3.6}$$

3.7. Example. (Example 1.1.12 of [5]) For the semi-group $\mathbb{N}^n \subset \mathbb{Z}^n$, we have

$$\mathbb{K}[\mathbb{N}^n] = \mathbb{K}[x_1, \dots, x_n],$$

with each $x_i = \chi^{e_i}$, where e_i is the standard basis vector.

$\diamondsuit$

3.8. Proposition. *(Proposition 1.1.14 in [5]) If $S = \mathbb{N}\mathcal{A}$ is an affine semi-group, then $\mathbb{K}[S] \cong \mathbb{K}[x_1, \dots, x_n]/\mathbb{I}(Y_{\mathcal{A}})$. Hence $\mathbb{K}[S]$ is the coordinate ring corresponding to the toric variety $Y_{\mathcal{A}}$.*

3.9. Notation. If Y is an affine variety we use the notation $\mathbb{K}[Y]$ as its coordinate ring where $\mathbb{K}[Y] = \mathbb{K}[x_1, \ldots, x_n]/\mathbb{I}(Y)$. Since varieties with isomorphic coordinate rings are isomorphic(see section 1.0 of [5]), we use the notation $Y = \operatorname{Spec}(\mathbb{K}[Y])$. There is an explicit construction of $\operatorname{Spec}(R)$ for a ring R, but for now we use this as notation to emphasize the relationship between the ring and the variety. For the toric variety $Y_{\mathcal{A}}$ above, we have $Y_{\mathcal{A}} = \operatorname{Spec}(\mathbb{K}[S])$.

3. The Affine Toric Variety of a Rational Polyhedral Cone.

3.10. Definition. (Definition 1.2.1 of [5]) Let $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ be a pair of dual vector spaces (for instance we could take $N_{\mathbb{R}} = N_{\mathbb{R}} = \mathbb{R}^n$, viewed as dual spaces via the usual inner product). A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \operatorname{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \geq 0 \right\},$$

where $S \subset N_{\mathbb{R}}$ is a finite subset. Let $m \in \sigma^{\vee}$ and define

$$H_m = \{u \in N_{\mathbb{R}} : \langle m, u \rangle = 0\},$$

called the hyperplane defined by m. A *face* τ of σ, written $\tau \prec \sigma$, is given by $\tau = \sigma \cap H_m$. We say that σ is *strongly convex* if $\{0\}$ is one of its faces (so that it is a cone emanating from the origin).

Given a convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$, its *dual cone* is defined to be

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

We are interested in cones which are generated by lattice points.

3.11. Definition. (Definition 1.2.14 of [5]) Let N and M be dual lattices with associated dual vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ (for instance we use $\mathbb{Z}^n$ as the lattice of $\mathbb{R}^n$). A *rational convex polyhedral cone* is a convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ such that $\sigma = \operatorname{Cone}(S)$ for some finite set $S \subset N$.

The usefulness of rational convex polyhedral cones is the following:

3.12. Proposition. *(Proposition 1.2.17 in [5]) Let $\sigma \subset N_{\mathbb{R}}$ be a rational convex polyhedral cone. Define the lattice points*

$$S_{\sigma} = \sigma^{\vee} \cap M \subset M.$$

Then S_{σ} is an affine semi-group.

Hence, starting from a rational convex polyhedral cone σ, we produce an affine semi-group by Proposition 3.12, then apply Proposition 3.8 to produce a toric variety $U_{\sigma} = \operatorname{Spec}(\mathbb{K}[S_{\sigma}])$.

4. The Toric Variety of a Fan.

We have constructed an affine toric variety from a rational convex polyhedral cone. We also saw that we can glue together affine varieties into an abstract variety. In this section we discuss the construction of an abstract toric variety from gluing together the affine toric varieties constructed from cones in the previous section.

3.13. Definition. (Definition 3.1.2 of [5]) A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones such that

(1) Every cone $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(2) For every $\sigma \in \Sigma$, each face of σ is also in Σ.

(3) For every pair $\sigma_1, \sigma_2 \in \Sigma$, their intersection is a face of both σ_1 and σ_2.

The conditions on the cones in a fan Σ are exactly what is necessary to satisfy the gluing conditions of Definition 3.1 to construct an abstract variety from the affine varieties $U_\sigma = \mathrm{Spec}(\mathbb{K}[S_\sigma])$. We denote the variety associated to the fan Σ as X_Σ (the details can be found in 3.1 of [5]).

Before giving the main result of this section, we have a series of definitions in order to understand the hypothesis of the theorem. Recall the notation from the definition of an abstract variety. We had affine varieties V_α, and homeomorphisms $h_\alpha\colon V_\alpha \to U_\alpha$ where the Zariski open sets U_α covered X.

3.14. Definition. (Definition 3.0.1 and 3.0.9 of [5]) Let V be an affine variety and $U \subset V$ a Zariski open subset. A map $\phi\colon U \to \mathbb{K}$ is called *regular* if for all $p \in U$ there exists $f_p \in \mathbb{K}[V]$ such that $p \in V_{f_p}$ and $\phi\big|_{V_{f_p}} \in \mathbb{K}[V]_{f_p}$, where $V_{f_p} = V \smallsetminus \mathbb{V}(f_p)$. In other words, there is a Zariski open set V_{f_p} such that $\phi = g/f_p^n$ for some $g \in \mathbb{K}[V]$ so that ϕ is locally a rational function.

Let U be an open subset of an abstract variety X and let $W_\alpha = h_\alpha^{-1}(U \cap U_\alpha) \subset V_\alpha$. A function $\phi\colon U \to \mathbb{K}$ is called *regular* if

$$\phi \circ h_\alpha\big|_{W_\alpha}\colon W_\alpha \to \mathbb{K}$$

is regular for all α.

Given an abstract variety X, a point $p \in X$, and neighbourhoods U_1 and U_2 of p, two regular functions f_1 and f_2 are *equivalent at* p, written $f_1 \sim f_2$, if there exists a neighbourhood $U \subset U_1 \cap U_2$ of p such that $f_1\big|_U = f_2\big|_U$. The *local ring of* X *and* p is defined to be

$$\mathcal{O}_{X,p} = \{f\colon U \to \mathbb{K} : U \text{ is a neighbourhood of } p \text{ and } f \text{ is regular}\}/\sim .$$

3.15. Definition. (See Section 1.0 and Definition 3.0.10 of [5]) Let R be an integral domain and $\mathbb{F}$ its field of fractions. R is called *normal* if every element of $\mathbb{F}$ which is the root of a monic polynomial in $R[x]$ lies in R (for example fields, unique factorization domains, and local rings are normal).

An abstract variety X is called *normal* if it is irreducible and the local rings $\mathcal{O}_{X,p}$ are normal for every $p \in X$.

We also need a condition on the topology of the variety.

3.16. Definition. (Definition 3.0.16 of [5]) A variety X is called *separated* if the image of the diagonal map

$$\Delta\colon X \to X \times X, \quad p \mapsto (p, p)$$

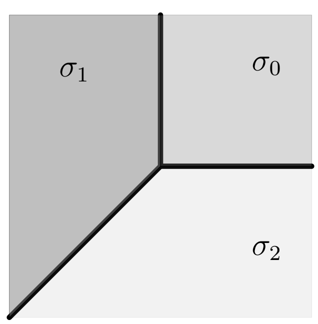
is Zariski closed. An equivalent definition when $\mathbb{K} = \mathbb{C}$ (Theorem 3.0.17 of [5]) is that X is Hausdorff in the classical topology.

We are now ready to state the main result from this section.

3.17. Theorem. *(Theorem 3.1.5 of [5]) The variety X_Σ associated to a fan Σ is a normal separated toric variety. Conversely, for every normal separated toric variety X with torus T_N (N the character lattice of T), there is a fan Σ in $N_\mathbb{R}$ such that $X \cong X_\Sigma$.*

3.18. Example. (Example 3.1.9 from [5]).

Let $N_\mathbb{R} = \mathbb{R}^2$ and $N = \mathbb{Z}^2$ with standard basis e_1, e_2. Then consider the fan Σ, pictured below.



Part (a) of Proposition 1.2.8 in [5] implies the dual cone is generated by the facet normals of the cones. In our case the dual cones of σ_0, σ_1, and σ_2 are generated by the normal vectors to the the rays which bound them.

For σ_0 we have

$$\sigma_0^\vee = \mathrm{Cone}(e_1, e_2).$$

This gives the semi-group $S_{\sigma_0} = \{(a, b) : a, b \in \mathbb{N}\} = \mathbb{N}^2$, and hence by 3.6 get the algebra $\mathbb{K}[S_{\sigma_0}] = \mathbb{K}[x, y]$ which gives

$$U_{\sigma_0} = \mathrm{Spec}(\mathbb{K}[x, y]).$$

For σ_1, we have

$$\sigma_1^\vee = \mathrm{Cone}(-e_1, -e_1 + e_2),$$

giving the semi-group

$$S_{\sigma_1} = \{(-a + b, b) : a, b \in \mathbb{N}\} = \mathbb{N}\{(-1, 0), (-1, 1)\}.$$

Hence we have the algebra

$$\mathbb{K}[S_{\sigma_1}] = \mathbb{K}[x^{-1}, x^{-1}y].$$

and the affine variety

$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{K}[x^{-1}, x^{-1}y])$$

Similarly,

$$U_{\sigma_2} = \mathrm{Spec}(\mathbb{K}[xy^{-1}, y^{-1}]).$$

Proposition 3.1.3 in [5] shows that if $\tau = \sigma_1 \cap \sigma_2$, then $S_\tau = S_{\sigma_1} + S_{\sigma_2}$. This provides us the gluing data necessary to construct the toric variety. Recall a face $\tau \prec \sigma$ is of the form $\tau = \sigma \cap H_m$. Proposition 1.3.16 of [5] shows that $\mathbb{K}[S_\tau]$ is the localization, $\mathbb{K}[S_\sigma]_{\chi^m}$. If $\tau = \sigma_1 \cap \sigma_2$, then we have $\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m$ for some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$. Hence we have

$$(U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}}.$$

This gives the gluing data on the coordinate rings,

$$\begin{aligned} g_{10}^* &\colon \mathbb{K}[x, y]_x \cong \mathbb{K}[x^{-1}, x^{-1}y]_{x^{-1}} \\ g_{20}^* &\colon \mathbb{K}[x, y]_y \cong \mathbb{K}[xy^{-1}, y^{-1}]_{y^{-1}} \\ g_{21}^* &\colon \mathbb{K}[x^{-1}, x^{-1}y]_{x^{-1}y} \cong \mathbb{K}[xy^{-1}, y^{-1}]_{xy^{-1}} \end{aligned}$$

Using homogeneous coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}^2$, then the map

$$x \mapsto x_1/x_0, \quad y \mapsto x_2/x_0$$

identifies the standard covering U_i of $\mathbb{P}^2$ with U_{σ_i}. For example,

$$U_{\sigma_0} \cong \mathrm{Spec}(\mathbb{K}[x, y]) \cong \mathrm{Spec}(\mathbb{K}[x_1/x_0, x_2/x_0]) \cong U_0$$

and similarly

$$U_{\sigma_1} \cong \mathrm{Spec}(\mathbb{K}[x^{-1}x^{-1}y]) \cong \mathrm{Spec}(\mathbb{K}[x_0/x_1, x_2/x_1]) \cong U_1.$$

From the gluing data, we have

$$g_{10}^* \colon \mathbb{K}[x, y]_x \cong \mathbb{K}[x^{-1}, x^{-1}y]_{x^{-1}}$$

which corresponds to the gluing data

$$g_{10}^* \colon \mathbb{K}[x_1/x_0, x_2/x_0]_{x_1/x_0} \cong \mathbb{K}[x_0/x_1, x_2/x_1]_{x_0/x_1}$$

from $\mathbb{P}^2$. Hence the variety X_Σ is $\mathbb{P}^2$.

$\Diamond$

5. The Total Coordinate Ring and Irrelevant Ideal

In this section we discuss the correspondence between toric varieties and ideals, generalizing the correspondence discussed in Section 1 of Chapter 2.

Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R}$. Similar to the quotient construction of $\mathbb{P}^n$, we wish to write X_Σ as

$$X_\Sigma = (\mathbb{K}^r \smallsetminus Z)/G$$

for some exceptional set Z and group action G.

Let $\Sigma(1)$ denote the set of one dimensional cones in Σ (the rays). Then we define the *total coordinate ring of* X_Σ to be the polynomial ring with one variable for each ray $\rho \in \Sigma(1)$,

$$S = \mathbb{K}[x_\rho : \rho \in \Sigma(1)].$$

For each $\sigma \in \Sigma$, let $\sigma(1)$ be the set of rays contained in σ, and define the monomial

$$x^{\hat{\sigma}} = \prod_{\rho \in \Sigma(1) \smallsetminus \sigma(1)} x_\rho.$$

Now we define the *irrelevant ideal of* X_Σ to be

$$B = \langle x^{\hat{\sigma}} : \sigma \in \Sigma \rangle.$$

Note that whenever we have a face $\tau \prec \sigma$, $x^{\hat{\sigma}}$ divides $x^{\hat{\tau}}$, so the irrelevant ideal is generated by the monomials corresponding to maximal cones. If we denote the set of maximal cones by $\Sigma_{\max}$, then we have

$$B = \langle x^{\hat{\sigma}} : \sigma \in \Sigma_{\max} \rangle.$$

Now we take the affine set $\mathbb{K}^{|\Sigma(1)|}$ and the exceptional set $Z(\Sigma) = \mathbb{V}(B)$. The group G in the quotient construction relies on the class group of X_Σ.

3.19. Definition. (See Section 4.0 of [5]) A *discrete valuation* on a field $\mathbb{K}$ is a group homomorphism $\nu \colon \mathbb{K}^* \to \mathbb{Z}$, which is surjective and satisfies

$$\nu(x + y) \geq \min(\nu(x), \nu(y)),$$

whenever $x, y, x + y \in \mathbb{K}^*$. The corresponding *discrete valuation ring* is

$$R = \{x \in \mathbb{K}^* : \nu(x) \geq 0\} \cup \{0\}.$$

Let X be an irreducible variety. A *prime divisor of* X is an irreducible sub-variety $D \subset X$ of co-dimension 1. We define $\operatorname{Div}(X)$ to be the free abelian group generated by prime divisors $D \subset X$. Elements of $\operatorname{Div}(X)$ are called *Weil divisors*.

If X is normal and D is a prime divisor of X, we get a discrete valuation ring

$$\mathcal{O}_{X,D} = \{\phi \in \mathbb{K}(X) \colon \phi \text{ is defined on } U \subset X \text{ open with } U \cap D \neq \emptyset\},$$

with a discrete valuation

$$\nu_D \colon \mathbb{K}(X)^* \to \mathbb{Z}.$$

If X is normal and $f \in \mathbb{K}^*(X)$, then $\nu_D(f)$ is zero for all but finitely many prime divisors $D \subset X$ (Lemma 4.09 of [5]). This allows us to define the *divisor of* $f \in \mathbb{K}(X)^*$,

$$\operatorname{Div}(f) = \sum_{D \subset X} \nu_D(f) D.$$

$\operatorname{Div}(f)$ is called a *principle divisor*, and the set of all principle divisors is denoted $\operatorname{Div}_0(X)$.

Now we have enough to define the class group,

3.20. Definition. (Definition 4.0.13 of [5]) Let X be a normal variety. The *class group of* X is

$$\mathrm{Cl(X)} = \operatorname{Div}(X)/\operatorname{Div}_0(X)$$

The group G involved in the quotient construction is

$$G = \operatorname{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{K}^*).$$

For more details on where this group comes from, see Section 5.1 of [5]. Theorem 5.1.10 of [5] then gives $X_\Sigma \cong ((\mathbb{K}^{|\Sigma(1)|}) \smallsetminus Z(\Sigma))/G$.

A nice feature of the total coordinate ring is that it is graded by the class group. Each ray $\rho \in \Sigma(1)$ corresponds to a co-dimension 1 orbit of the torus $T_N \subset X_\Sigma$ by Theorem 3.2.6 in [5], whose closure is a prime divisor D_ρ. In fact Theorem 4.1.3 in [5] shows that the classes of these D_ρ in $\mathrm{Cl}(X)$ form a generating set.

For $a = (a_\rho) \in \mathbb{Z}^{|\Sigma(1)|}$, and a monomial $x^a = \prod x_\rho^{a_\rho}$, we define the degree of x^a to be

$$\deg(x^a) = \sum_\rho a_\rho D_\rho \in \mathrm{Cl}(X),$$

so that

$$\deg(x_\rho) = D_\rho.$$

For $\beta \in \mathrm{Cl}(X)$, we say that $f \in S_\beta$ is homogeneous of degree β.

With one more definition, we will have enough algebraic structure to give a correspondence for toric varieties and ideals.

3.21. Definition. (Definitions 1.2.16 and 3.1.18 of [5]) A cone σ is called *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

A toric variety X_Σ is called *simplicial* if every $\sigma \in \Sigma$ is simplicial.

In the following proposition by homogeneous we mean homogeneous with respect to the grading induced by the class group $CL(X_\Sigma)$.

3.22. Proposition. *(Proposition* 5.2.7 *of [5]) Let* X_Σ *be a simplicial toric variety. There is a bijective correspondence*

$$\{\textit{Closed sub-varieties of } X_\Sigma\} \iff \{\textit{Radical homogeneous ideals } I \subset B \subset S\}$$

Recall that an ideal $I \subset S$ is called $B-$saturated if $(I : B^\infty) = I$. When the toric variety X_Σ is smooth (for instance projective space or a product of projective spaces), Corollary 3.8 in [**COX**] gives another correspondence:

$$\{\text{Closed sub-varieties of } X_\Sigma\} \iff \{B-\text{saturated radical homogeneous ideals } I \subset S\}.$$

The relationship between these two sets of ideals is given by the bijective maps

$$I \text{ radical, homogeneous, } B \text{ saturated} \mapsto I \cap B,$$

and

$$J \text{ radical, homogeneous, contained in } B \mapsto (J : B^\infty).$$

CHAPTER 4

Virtual Resolutions

We could now apply the techniques of minimal free resolutions discussed earlier to the ideals associated to closed sub-varieties of a toric variety X_Σ. The principal example we are interested in is when X_Σ is a product of projective spaces, $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. In this case, in turns out that the minimal free resolutions are larger than is necessary to capture much of the geometric information. In [2], virtual resolutions are defined and shown to be a useful analogue to minimal free resolutions. The idea is to look at a free resolution of modules that when "sheafified" give a resolution in the sense of sheaves. This allows a certain amount of homology in the complex, and often leads to shorter sequences than the minimal free resolution.

1. The Definition of a Virtual Resolution

In order to define a virtual resolution, we first need to briefly discuss sheaves.

4.1. Definition. (See Section 2.1 of [8]) Let X be a topological space. A *presheaf* of abelian groups (or rings, or modules, etc.) on X is a collection of abelian groups $\mathcal{F}$ satisfying the following:

(1) For every open set $U \subset X$ there is an abelian group $\mathcal{F}(U)$,

(2) For every inclusion $V \subset U$ of open sets, there is a morphism
$$\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V),$$

(3) $\mathcal{F}(\emptyset) = 0$,

(4) ρ_{UU} is the identity map,

(5) If $W \subset V \subset U$ are three open sets then $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$.

Hence a presheaf is a collection of abelian groups (rings, modules, etc) sitting above each open set of X along with maps ρ_{UV} which "restrict" the group above U to the group above V. the group $\mathcal{F}(U)$ is called the *section* of $\mathcal{F}$ above U

A sheaf is a presheaf with some extra local conditions on the sections.

4.2. Definition. (See Section 2.1 of [8]) A *sheaf* $\mathcal{F}$ on X is a presheaf satisfying the following:
Let U be an open subset of X and $\{V_i\}$ an open covering of U

(1) If $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = 0$ for all i, then $s = 0$.

(2) If $s_i \in \mathcal{F}(V_i)$ such that for each i, j we have $\rho_{UV_i \cap V_j}(s_i) = \rho_{UV_i \cap V_j}(s_j)$, then there is $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = s_i$ for each i.

4.3. Example. (see section 3 of [5]) Recall from definition 3.14 what it means for a function $\phi\colon U \to K$ to be a regular function on an open set U of an abstract variety X.

Let $\mathcal{O}_X(U) = \{\phi\colon U \to K : \phi \text{ is regular}\}$. Then $\mathcal{O}_X$ is a sheaf of rings on X. $\mathcal{O}_X$ is called the *structure sheaf* of X. Hence an abstract variety is a ringed space $(X, \mathcal{O}_X)$ with a finite open covering U_α such that $(U_\alpha, \mathcal{O}_X\big|_{U_\alpha})$ is isomorphic to the ringed space $(V_\alpha, \mathcal{O}_{V_\alpha})$ for an affine variety V_α (see section 2.2 of [8] for more on ringed spaces and schemes). $\Diamond$

4.4. Definition. (See Section 2.1 of [8]) A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ which commutes with the restriction maps, i.e. the following diagram commutes whenever $V \subset U \subset X$ are open,

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle \rho_{UV}} & & \Big\downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves, we define the *presheaf kernel* and the *presheaf image* of φ to be

$$U \mapsto \ker(\varphi(U)), \quad U \mapsto \operatorname{im}(\varphi(U)).$$

When $\mathcal{F}$ and $\mathcal{G}$ are sheaves, the presheaf kernel is in fact a sheaf, however the image may not be.

A similar definition can be made for the quotient of two (pre)sheaves. If $\mathcal{F}(U)$ is a subgroup (subring, submodule, etc.) of $\mathcal{G}(U)$ for all U, then we define the *presheaf quotient* to be

$$U \mapsto \mathcal{F}(U)/\mathcal{G}(U).$$

Again, the presheaf quotient may not be a sheaf.

However, proposition 1.2 in chapter 2 of [8] shows that for every presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ which is unique up to a unique isomorphism. When $\mathcal{F}$ is a sheaf, $\mathcal{F} \cong \mathcal{F}^+$ by this unique isomorphism. We define the *image* and *quotient* sheaves to be the unique sheaves associated to the presheaf image and the presheaf quotient respectively.

Now let X_Σ be the toric variety associated to a fan Σ, let S be the total coordinate ring (graded by the class group), and let B be the irrelevant ideal. Let M be a graded $S-$module. Then proposition 5.3.3 from [5] shows that there exists a sheaf of modules $\widetilde{M}$ on X_Σ whose sections above the open sets U_σ are given by the degree zero elements of the localized module $M_{x^{\hat\sigma}}$,

$$\widetilde{M}(U_\sigma) = (M_{x^{\hat\sigma}})_0.$$

This gives us a functor from $S-$modules to quasi-coherent sheaves on X_Σ. Proposition 3.1 of [3] shows that this is an exact functor, so that whenever we have an exact sequence

$$0 \to M \to N \to N/M \to 0,$$

we get an exact sequence of sheaves

$$0 \to \widetilde{M} \to \widetilde{N} \to \widetilde{(N/M)} \to 0.$$

4.5. Definition. (Definition 1.1 in [2]) Let X_Σ be a toric variety, S the total coordinate ring, and B the irrelevant ideal. A *virtual resolution* of a graded $S-$module M is a complex of free $S-$modules

$$F : F_0 \leftarrow F_1 \leftarrow F_2 \ldots$$

such that the associated complex of sheaves

$$\widetilde{F} : \widetilde{F_0} \leftarrow \widetilde{F_1} \leftarrow \widetilde{F_2} \ldots$$

is a locally free resolution of the sheaf $\widetilde{M}$. That is, the complex $\widetilde{F}$ is exact in all indices greater than 0 and the sheaf quotient $\widetilde{F_0}/\operatorname{im}(\widetilde{\varphi}_1)$ is isomorphic to $\widetilde{M}$.

2. An Algebraic Condition for a Virtual Resolution

When X_Σ is smooth, we have a nice algebraic definition of a virtual resolution. First, we discuss the algebraic condition on the complex F to guarantee that the complex $\tilde{F}$ is exact at indices $i \geq 1$. This relies on a few things. The first, which was already mentioned, is that the functor taking modules to sheaves is exact.

Exercise 1.6 from chapter 2 of [8] shows that whenever $\mathcal{F}'$ is a sub-sheaf of $\mathcal{F}$, we get an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0,$$

and whenever we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

then $\mathcal{F}'$ is isomorphic to a sub-sheaf of $\mathcal{F}$ and $\mathcal{F}''$ is isomorphic to the quotient sheaf $\mathcal{F}/\mathcal{F}'$.

We also need to discuss the construction of $\widetilde{M}$ on an affine variety Spec(R). We previously used Spec(R) just as notation to emphasize the relationship between the coordinate ring and the variety. However, there is an explicit construction of Spec(R) from the ring R. As a set, it consists of the prime ideals of R. For a detailed construction of the topology and structure sheaf, see section 2 of chapter 2 in [8]. We then define $\widetilde{M}(U)$ to be the set of all functions $s\colon U \to \coprod_{p \in U} M_p$, with $s(p) \in M_p$ and s is locally a fraction. That is, for each $p \in U$ there must exist a neighbourhood $V \subset U$ containing p, and elements $m \in M$ and $f \in R$ such that for all $q \in V$, we have $f \notin q$ and $s(q) = m/f \in M_q$.

4.6. Lemma. *Let*

$$F : F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \cdots$$

be a complex of free $S-$modules and

$$\widetilde{F} : \widetilde{F_0} \xleftarrow{\widetilde{\varphi_0}} \widetilde{F_1} \xleftarrow{\widetilde{\varphi_2}} \widetilde{F_2} \cdots$$

the associated complex of sheaves on X_Σ. Then $\widetilde{H_i(F)} \cong H_i(\widetilde{F})$.

Proof. We have the exact sequence

$$0 \to \operatorname{im}(\varphi_{i+1}) \to \ker(\varphi_i) \to H_i(F) \to 0,$$

which gives the exact sequence

$$0 \to \widetilde{\operatorname{im}(\varphi_{i+1})} \to \widetilde{\ker(\varphi_i)} \to \widetilde{H_i(F)} \to 0.$$

If we can show that the functor taking modules to sheaves commutes with the image and kernel, then we will have

$$\widetilde{H_i(F)} \cong H_i(\widetilde{F})$$

by the above mentioned exercise.

Fortunately, by proposition-definition 1.2 in chapter 2 of [8], we only need to show that the pre sheaves agree on every open set $U \subset X_\Sigma$, and we do this by looking at the affine pieces.

For a polynomial ring R, We can show that $\operatorname{im}(\widetilde{\varphi})(U) = \widetilde{\operatorname{im}(\varphi)}(U)$ for any map $\varphi\colon M \to N$ of $R-$modules where the sheaves are constructed on $\operatorname{Spec}(R)$ as above. For each prime ideal p, φ gives a localized map $\varphi\colon M_p \to N_p$ given by

$$\varphi(m/f) = \varphi(m)/f.$$

The induced map on sections, $\widetilde{\varphi}(U)\colon \widetilde{M}(U) \to \widetilde{N}(U)$, is given by

$$\widetilde{\varphi}(U)(s) = \varphi \circ s\colon U \to \coprod_{p \in U} N_p.$$

It is clear that $\varphi \circ s\colon U \to \coprod_{p\in U}(\operatorname{im}(\varphi))_p \subset \coprod_{p\in U} N_p$, so that one inclusion is immediate.

On the other hand, suppose that we have $t \in \widetilde{\operatorname{im}(\varphi)}(U)$. Then for every $p \in U$, $t(p) = \varphi(m)/f$ for some $m \in M_p$ and $f \notin p$. We then define $s\colon U \to \coprod_{p\in U} M_p$ by $s(p) = m/f$. By the local nature of t, we also have that s is locally a fraction and hence in $\widetilde{M}(U)$. Then $t = \varphi \circ s$, showing that $t \in \operatorname{im}(\widetilde{\varphi})(U)$.

Hence, for every section we have $\widetilde{\operatorname{im}(\varphi)}(U) = \operatorname{im}(\widetilde{\varphi})(U)$. This shows that they are equal as presheaves, implying they are equal as sheaves. A similar argument shows that $\ker(\widetilde{\varphi}) = \widetilde{\ker(\varphi)}$.

We then get the exact sequence

$$0 \to \operatorname{im}(\widetilde{\varphi_{i+1}}) \to \ker(\widetilde{\varphi}_i) \to \widetilde{H_i(F)} \to 0,$$

showing that $H_i(\widetilde{F}) = \widetilde{H_i(F)}$ by exercise 1.6 of [8] mentioned above.

□

Corollary 3.6 from [3] shows that $\widetilde{M} = 0$ if and only if there is some power t such that $B^t M = 0$, hence we get the following:

4.7. Lemma. *The complex of sheaves $\widetilde{F}$ is exact if and only if for each $i > 0$ there is some power t such that $B^t H_i(F) = 0$.*

Proof. The complex is exact if and only if $H_i(\widetilde{F}) = 0$ for each $i > 0$. Applying Lemma 4.6 and Corollary 3.6 from [3] shows this is equivalent to there being a power of t such that $B^t H_i(F) = 0$. □

Now we give a condition on modules which will guarantee that they give the same sheaves on X_Σ. For an $S-$module M, we define the following submodule:

$$\Gamma_B(M) = \{m \in M : B^t m = 0 \text{ for some } t \in \mathbb{N}\}.$$

Exercise 13 from section 10.1 of [6] shows that this is indeed a submodule.

4.8. Lemma. *Let M and N be two finitely generated $S-$modules. If*

$$M/\Gamma_B(M) \cong N/\Gamma_B(N),$$

then

$$\widetilde{M} \cong \widetilde{N}.$$

Proof. We have the exact sequence

$$0 \to \Gamma_B(M) \to M \to M/\Gamma_B(M) \to 0,$$

which sheafifies to the exact sequence

$$0 \to \widetilde{\Gamma_B(M)} \to \widetilde{M} \to \widetilde{M/\Gamma_B(M)} \to 0.$$

Since S is Noetherian, $\Gamma_B(M)$ is finitely generated. Let $m_1, \ldots, m_k$ be a generating set, and $t_1, \ldots, t_k$ be the powers such that $B^{t_i} m_i = 0$. Then setting $t = \max\{t_1, \ldots, t_k\}$ gives $B^t m_i = 0$ for all i, and hence $B^t \Gamma_B(M) = 0$. This gives us the exact sequence

$$0 \to \widetilde{M} \to \widetilde{M/\Gamma_B(M)} \to 0,$$

showing that the sheaves are isomorphic. Similarly, we have $\widetilde{N} \cong \widetilde{N/\Gamma_B(N)}$. Then by our hypothesis ($M/\Gamma_B(M) \cong N/\Gamma_B(N)$), we have $\widetilde{M/\Gamma_B(M)} \cong \widetilde{N/\Gamma_B(N)}$. This gives

$$\widetilde{M} \cong \widetilde{M/\Gamma_B(M)} \cong \widetilde{N/\Gamma_B(N)} \cong \widetilde{N}.$$

□

This leads to the following algebraic condition for a complex to be a virtual resolution.

4.9. Theorem. *Let M be a finitely generated $S-$module and*

$$F : F_0 \leftarrow F_1 \leftarrow F_2 \cdots$$

a complex of free $S-$modules satisfying:

(1) *For each $i > 0$ there is some power t such that $B^t H_i(F) = 0$,*
(2) *$H_0(F)/\Gamma_B(H_0(F)) \cong M/\Gamma_B(M)$.*

Then F is a virtual resolution of M.

Proof. The first condition shows that the sequence of sheaves is exact by Lemma 4.7, and the second condition shows that

$$\widetilde{M} \cong \widetilde{H_0(F)} \cong H_0(\widetilde{F}),$$

by Lemma 4.6 and Lemma 4.8. □

We will usually be concerned with the sheaves corresponding to $B-$saturated radical homogeneous ideals, since these correspond to the closed sub-schemes of X_Σ. Moreover our main methods of producing virtual resolutions come from the minimal free resolutions of such ideals. In this case $F_0 = S$, and we state this as a special case.

4.10. Theorem. *Suppose that*

$$F : S \xleftarrow{\varphi_1} F_1 \leftarrow F_2 \cdots$$

is a complex of free $S-$modules satisfying

(1) *For each $i > 0$ there is some power t such that $B^t H_i(F) = 0$,*
(2) *$(\mathrm{im}(\varphi_1) : B^\infty) = (I : B^\infty)$.*

Then F is a virtual resolution of S/I.

Proof. We apply Theorem 4.9. The first condition is satisfied by our hypothesis on F. For the module S/I, we have

$$\Gamma_B(S/I) = \{f + I : \text{ there is some power } t \text{ such that } B^t f \in I\} = (I : B^\infty)/I.$$

The Third Isomorphism Theorem (Theorem 4 of section 10.2 in [6]) gives

$$(S/I)/(\Gamma_B(S/I)) \cong S/(I : B^\infty).$$

Hence condition (2) of Theorem 4.9 reduces to

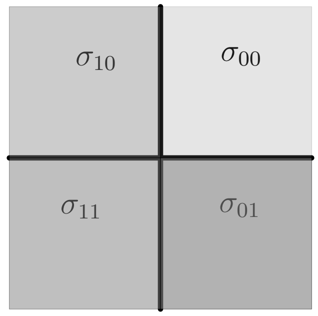
$$S/(\mathrm{im}(\varphi_1) : B^\infty) \cong S/(I : B^\infty),$$

which is satisfied by our second hypothesis.

□

3. Virtual Resolutions of Points in a Product of Projective Spaces.

First, let us consider the simplest example of a product of projective spaces, $\mathbb{P}^1 \times \mathbb{P}^1$(see Example 3.1.12 in [5]. Consider the following fan:



This gives us the affine open cover

$$U_{00} = \operatorname{Spec}(K[S_{\sigma_{00}}]) \cong \operatorname{Spec}(K[x, y]),$$

$$U_{10} = \operatorname{Spec}(K[S_{\sigma_{10}}]) \cong \operatorname{Spec}(K[x^{-1}, y]),$$

$$U_{11} = \operatorname{Spec}(K[S_{\sigma_{11}}]) \cong \operatorname{Spec}(K[x^{-1}, y^{-1}]),$$

$$U_{01} = \operatorname{Spec}(K[S_{\sigma_{01}}]) \cong \operatorname{Spec}(K[x, y^{-1}]).$$

Now if U_0 and U_1 are the standard affine open sets covering $\mathbb{P}^1$ with coordinate rings $K[x]$ and $K[x^{-1}]$, then we see that $U_i \times U_j$ has the same coordinate ring as $U_{\sigma_{ij}}$ above. Hence $U_{\sigma_{ij}} \cong U_i \times U_j$. The gluing of these affine varieties gives us $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, the total coordinate ring is in 4 variables,

$$S = K[x_0, x_1, y_0, y_1],$$

where the variables x_0, x_1 correspond to the horizontal rays, and the variables y_0, y_1 correspond to the vertical rays. The irrelevant ideal is then

$$B = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1\rangle = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle.$$

More generally (see example 2.4.8 and 3.1.12 in [5]), for $\mathbb{P}^n \times \mathbb{P}^m$, we have total coordinate ring

$$K[x_0, \ldots, x_n, y_0, \ldots, y_m],$$

and the irrelevant ideal is

$$B = \langle x_0, \ldots, x_n\rangle \cap \langle y_0, \ldots, y_m\rangle.$$

Example 2.4.8 and Proposition 2.4.9 in [5] shows that this generalizes to multiple factors.

4.11. Notation. From now on, we let

$$X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

The total coordinate ring of X is

$$S = K[x_{01}, \ldots, x_{n_1 1}, \ldots, x_{0r}, \ldots, x_{n_r r}]$$

which is graded by $\mathbb{Z}^r$ with $\deg(x_{ij}) = e_j$, and the irrelevant ideal is

$$B = \bigcap_{j=1}^{r} \langle x_{0j}, \ldots, x_{n_j j} \rangle.$$

Note that if I is $B-$saturated, then any minimal free resolution of S/I is a virtual resolution. Indeed, if F is the minimal free resolution, then $H_i(F) = 0$ for all $i \geq 1$, and $\mathrm{im}(\varphi_1) = I = (I : B^\infty)$, so the definition of a virtual resolution is a generalization of the minimal free resolution of an ideal.

For $\mathbf{a} \in \mathbb{N}^r$, let $B^{\mathbf{a}} = \bigcap_{j=1}^{r} \langle x_{0j}, \ldots, x_{n_j j} \rangle^{a_i}$. The following theorem gives one method for constructing a virtual resolution of a set of points.

4.12. Theorem. *(Theorem 4.1 in [2])*
Let $Z \subset X$ be a set of points and I the corresponding $B-$saturated S ideal. There exists $\mathbf{a} \in \mathbb{N}^r$ with $a_r = 0$ and the other entries sufficiently positive such that the minimal free resolution of $S/(I \cap B^{\mathbf{a}})$ has length $n_1 + n_2 + \ldots + n_r$, and is a virtual resolution of S/I.

Proof. The proof that the minimal free resolution has the specified length is given in [2].

We show that the minimal free resolution of $S/(I \cap B^{\mathbf{a}})$ is a virtual resolution of S/I. Let the resolution be

$$F : F_0 \xleftarrow{\varphi_1} F_1 \leftarrow F_2 \cdots$$

Since the complex is exact, for each $i \geq 1$ $H_i(F) = 0$, so the first condition is satisfied trivially. Now since the resolution is minimal, we have $F_0 = S$ and $\mathrm{im}(\varphi_1) = I \cap B^{\mathbf{a}}$. Let n be the product of the non-zero a_i from the exponent vector $\mathbf{a}$. Then for any $g \in B$, $g^n \in B^{\mathbf{a}}$, so we see that $B^n \subset B^{\mathbf{a}}$. Then for any $f \in I$ we have $fB^n \subset I \cap B^{\mathbf{a}}$. Hence $I \subset (I \cap B^{\mathbf{a}} : B^\infty)$. The other inclusion follows easily, showing that $(\mathrm{im}(\varphi_1) : B^\infty) = I$. Hence the complex is a virtual resolution of S/I.

□

4. Examples

The following examples consider points in $X = \mathbb{P}^1 \times \mathbb{P}^1$. The total coordinate ring is $R = K[x_0, x_1, y_0, y_1]$ graded by $\mathbb{Z}^2$, and the irrelevant ideal is $\langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$.

4.13. Example. First, consider the point $[1:0] \times [1,0] \in \mathbb{P}^1 \times \mathbb{P}^1$. The associated $B-$saturated radical ideal is $I = \langle x_1, y_1 \rangle$. Using Macaulay 2, we calculate the minimal free resolution to be

$$S \xleftarrow{\begin{pmatrix} x_1 & y_1 \end{pmatrix}} \begin{matrix} S(-1,0) \\ \oplus \\ S(0,-1) \end{matrix} \xleftarrow{\begin{pmatrix} -y_1 \\ x_1 \end{pmatrix}} S(-1,-1) \leftarrow 0,$$

which has length two. Therefore, we could take $\mathbf{a} = (0,0)$ in Theorem 4.12 to see that this minimal free resolution is a virtual resolution of S/I of length 2.

A slightly more interesting example is to add the point $[0,1] \times [0,1]$. The ideal associated to the set of points

$$\{[1:0] \times [1,0], [0,1] \times [0,1]\}$$

is $I = \langle y_0 y_1, x_0 y_1, x_1 y_0, x_0 x_1 \rangle$, which has minimal free resolution of the form

$$S \xleftarrow{\varphi_1} \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1)^2 \\ \oplus \\ S(0,-2) \end{matrix} \xleftarrow{\varphi_2} \begin{matrix} S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \end{matrix} \xleftarrow{\varphi_3} S(-2,-2) \leftarrow 0,$$

where

$$\varphi_1 = \begin{pmatrix} x_0 x_1 & x_1 y_0 & x_0 y_1 & y_0 y_1 \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -y_0 & -y_1 & 0 & 0 \\ x_0 & 0 & 0 & -y_1 \\ 0 & x_1 & -y_0 & 0 \\ 0 & 0 & x_0 & x_1 \end{pmatrix},$$

and

$$\varphi_3 = \begin{pmatrix} y_1 \\ -y_0 \\ -x_1 \\ x_0 \end{pmatrix}.$$

We see this minimal free resolution is longer than the virtual resolution from Theorem 4.12. Take $\mathbf{a} = (1,0)$ and consider $I \cap B^{\mathbf{a}}$. The minimal free resolution is

$$S \xleftarrow{\varphi_1} \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1)^2 \end{matrix} \xleftarrow{\varphi_2} S(-2,-1)^2 \leftarrow 0,$$

where

$$\varphi_1 = \begin{pmatrix} x_0 x_1 & x_1 y_0 & x_0 y_1 \end{pmatrix},$$

and

$$\varphi_2 = \begin{pmatrix} -y_0 & -y_1 \\ x_0 & 0 \\ 0 & x_1 \end{pmatrix}.$$

We can see in this case that the minimal free resolution of $S/(I \cap B^{\mathbf{a}})$ is shorter and thinner than the minimal free resolution of S/I.

◊

The above example and Theorem 4.12 show how a virtual resolution can be constructed by eliminating the requirement that the $0'$th homology be S/I, however the complexes are still exact. The other possibility is to allow a limited amount of homology. Theorem 1.3 and Algorithm 3.4 of [2] give a method of producing virtual resolutions which are in general not exact. This method of producing virtual resolutions involves removing summands which are generated in large enough degree from a minimal free resolution. The algorithm is implemented in Macaulay 2 via the package "VirtualResolutions".

4.14. Example. (Example 4.3 of [10]) Consider the ideal of the set of points

$$\{[1:1]\times[1:4],[1:2]\times[1:5],[1:3]\times[1:6]\}.$$

Macaulay 2 calculates the $B-$saturation of the ideal to be

$$I = \langle 3x_0y_0 + x_1y_0 - x_0y_1, 120y_0^3 - 74y_0^2y_1 + 15y_0y_1^2 - y_1^3, 120x_1y_0^2 - 34x_1y_0y_1 - 2x_0y_1^2 + 3x_1y_1^2 \\ 40x_1^2y_0 + 6x_0^2y_1 - 13x_0x_1y_1 - 3x_1^2y_1, 6x_0^3 - 11x_0^2x_1 + 6x_0x_1^2 - x_1^3\rangle.$$

The minimal free resolution of S/I is calculated to be

$$S \leftarrow \begin{matrix} S(-1,-1) \\ \oplus \\ S(-3,0) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(0,-3) \end{matrix} \leftarrow \begin{matrix} S(-3,-1)^2 \\ \oplus \\ S(-2,-2)^2 \\ \oplus \\ S(-1,-3)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-2) \\ \oplus \\ S(-2,-3) \end{matrix} \leftarrow 0.$$

However, if we apply the algorithm and remove the summands which are generated in degree more than $(3,1)$, we get the virtual resolution

$$S \xleftarrow{\varphi_1} \begin{matrix} S(-1,-1) \\ \oplus \\ S(-3,0) \\ \oplus \\ S(-2,-1) \end{matrix} \xleftarrow{\varphi_2} S(-3,-1)^2 \leftarrow 0,$$

where

$$\varphi_1 = \begin{pmatrix} x_0y_0 + 1/3x_1y_0 - 1/3x_0y_1 \\ x_0^3 - 11/6x_0^2x_1 + x_0x_1^2 - 1/6x_1^3 \\ x_1^2y_0 + 3/20x_0^2y_1 - 13/40x_0x_1y_1 - 3/40x_1^2y_1 \end{pmatrix}^T,$$

and

$$\varphi_2 = \begin{pmatrix} -x_0^2 + 13/6x_0x_1 - 31/18x_1^2 & -x_1^2 \\ y_0 - 1/3y_1 & -3/20y_1 \\ 20/27x_1 & x_0 + 1/3x_1. \end{pmatrix}.$$

Macaulay 2 shows that $(\mathrm{im}(\varphi_1) : B^\infty) = I$ and that the complex is exact so it is indeed a virtual resolution of S/I. ◊

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