An Introduction to the Theory and Applications of Continued Fractions

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Abstract

This paper is the written component of my independent study on expressions of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{\ddots}}}}}$$

Such expressions are called *continued fractions*. In this paper, I give some of the basic definitions of continued fractions, a brief history of the subject, as well as proving a number of elementary theorems that utilize continued fractions. I conclude this paper by detailing my own project, that is, making a World Wide Web site devoted to continued fractions. This web site enables the casual or interested user to explore the subject of continued fractions. The web site contains not only a number of theorems, but interactive programs that I have written that use or compute continued fractions. The Uniform Resource Locator (URL) of the site is *http://www.calvin.edu/∼avtuyl52/confrac/*.

Forward

This paper has been written to fulfill one of the requirements for an Honors Degree from the mathematics department of Calvin College. This paper should be accessible to other undergraduate mathematics majors who are either at the junior or senior level. The subject of this report is continued fractions, a field of mathematics that is not commonly taught at Calvin but has the property of being easily understood by upper level mathematics majors.

I would like to thank all those who helped me by answering one or more of the many questions I came up against, especially those who answered some of my computer related questions. I would especially like to thank Dr. Thomas Jager for his suggestions and Dr. Paul Zwier for his willingness to help me on this project and providing me with much needed direction and insight.

Contents

1 Introduction and Basic Definitions

Consider the following example from first year calculus. Suppose that you are given the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots\}$. One of the natural questions one can ask about this sequence is whether the sum of all the elements is defined. That is, does the series

$$\sum_{i=0}^{\infty} \frac{1}{2^i}$$

converge to some real number, or does the series diverge? To determine this, we can ask whether the sequence $\{S_n\}$ converges to some number, where S_n is defined to be the partial sum

$$S_n = \sum_{i=0}^{n} \frac{1}{2^i}.$$

If the $\lim_{n\to\infty} S_n$ exists and is some s, then the sum is equal to s. We know from calculus that in our example the series does in fact converge. When we calculate the above series, we find that

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2.$$

To solve this problem we made use of the fact that our series is a geometric series, that is, successive terms are separated by a common ratio, in this case, $\frac{1}{2}$.

We can generalize the above example to the following. If we are given a sequence $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$, we can ask whether the sum

$$\sum_{i=1}^{\infty} a_i$$

is defined. To check if this is true, we merely have to determine if the sequence $\{S_n\}$, where S_n is defined to be

$$S_n = \sum_{i=1}^{n} a_i,$$

converges to some number s. If it does, the sum converges to this number.

We also know from calculus that we can ask whether the product of all the terms in a sequence converges or diverges, that is, is

$$\Pi_{i=1}^{\infty} a_i = a_1 a_2 a_3 \ldots$$

defined. To determine if this is true, we use a similar method of creating a sequence and asking whether that sequence converges to some number p. In this case, we form the sequence P_n where

$$P_n = \Pi_{i=1}^{n} a_i.$$

If the sequence $\{P_n\}$ converges to some number p, this is the value to which the product will also converge.

In both cases, we are curious about the infinite application of a binary function. In the first case, we want to know what happens when we add an infinite number of things together; in the second case, we want to know what happens when we multiply an infinite number of things together.

We can take this a step further and ask if it makes sense to divide and add an infinite number of times. That is, if we are given two sequences $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ and $\{b_1, b_2, b_3, \ldots, b_n, \ldots\}$, is the expression

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{\ddots}}}}}$$

defined. Expressions of this form were eventually given the name of *continued fractions*. Once again, by forming the sequence $\{C_n\}$, where

$$C_n = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\cfrac{\ddots}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}},$$

we can determine whether the continued fraction exists. It will do so if $\{C_n\}$ converges to some real number c. In searching for the answer to this question, the foundation for the field of continued fractions was laid.

In this paper, we will examine this field of mathematics devoted to continued fractions. We will begin with a historical overview of this discipline. Following this, we will prove some of the classical theorems that deal with these fractions. We will also see some of the uses of continued fractions, such as its use in determining which gears to use to create a desired gear ratio. Finally, I will discuss my own particular project that involved continued fractions, specifically, creating an introductory web site for continued fractions.

Before we do this, however, we need to go over some basic definitions and ideas that are used throughout this paper. Below are a number of basic terms and definitions connected with the field of continued fractions. They have been derived from [6], [7], and [9].

Definition 1 An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{\ddots}}}}}$$

is said to be a *continued fraction.* The values of $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ can be either real or complex values. There can be either an infinite or a finite number of terms. Note that if $b_n = 0$ for any n, then the continued fraction is finite.

Definition 2 A *simple continued fraction* is a continued fraction in which the value of $b_n = 1$ for all n, that is,

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{\ddots}}}}}$$

The value of a_n is a positive integer for all $n \geq 2$; a_1 can be any integer value, including 0. The above fraction is sometimes represented by $[a_1; a_2, a_3, \ldots]$. If the fraction is finite, we represent it by $[a_1; , a_2, \ldots, a_n]$.

Definition 3 The *terms* of a simple continued fraction refer to the values of $a_1, a_2, a_3, \ldots$. For example, a_4 is the fourth term. Sometimes, they are referred to as *partial quotients*.

Definition 4 - 5 A *finite simple continued fraction* is a simple continued fraction with only a finite number of terms. An *infinite simple continued fraction* is a simple continued fraction with an infinite number of terms.

Definition 6 The simple continued fraction $[a_1; a_2, a_3, \ldots, a_k]$, where k is such that $1 \leq k \leq n$, is called the k^{th} *convergent* of the simple continued fraction $[a_1; a_2, a_3, \ldots, a_n]$, or $[a_1; a_2, a_3, \ldots, a_n, \ldots]$. The k^{th} convergent is denoted by C_k. For example,

$$C_1 = 1,$$

$$C_2 = \frac{3}{2},$$

and

$$C_3 = \frac{10}{7}$$

are the three convergents of the simple continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{3}}.$$

Remark

It should be noted that throughout this paper the continued fractions that we are going to focus upon are simple continued fractions, that is, a continued fraction where $b_i = 1$ for all i, and a_i is a positive integer for all $i > 1$. The value of a_1 can be any integer value, including zero. We will examine both finite and infinite simple continued fractions.

2 History

To do mathematics, that is, in order to understand and to make contributions to this discipline, it is necessary to study its history. Unlike most other disciplines, mathematics is constantly building upon past discoveries. This is due to the nature of mathematics. Once something has been demonstrated conclusively to be true (or, for that matter, false) the case is considered closed. For example, the square root of two can always be shown to be irrational. Thus, those who wish to study a particular field of mathematics, whether it be statistics, abstract algebra, or continued fractions, will first need to study the field's past. In doing so, one is able to build upon past accomplishments rather than repeating them.

The origin of continued fractions is hard to pinpoint. This is due to the fact that we can find examples of these fractions throughout mathematics in the last 2000 years, but its true foundations were not laid until the late 1600's and early 1700's.

The origin of continued fractions is traditionally placed at the time of the creation of Euclid's Algorithm.[7] Euclid's Algorithm can be used to find the greatest common denominator (gcd) of two numbers. However, by algebraically manipulating the algorithm, one can derive the simple continued fraction of the rational $\frac{p}{q}$ as opposed to the gcd of p and q. This manipulation will be demonstrated later in the paper. It is doubtful whether Euclid or his contemporaries actually used this algorithm in such a manner. But due to its close relationship to continued fractions, the creation of Euclid's Algorithm signifies the initial development of continued fractions.

For more than a thousand years, any work that used continued fractions was restricted to specific examples. The Indian mathematician Āryabhata (d. 550 AD) used a continued fraction to solve a linear indeterminate equation (more will be discussed on this topic later in the paper).[7] However, Āryabhata did not generalize his method.

Throughout Greek and Arabic mathematical writing, we can find examples and traces of continued fractions.[7] But again, its use is limited to specific problems.

Two men from the city of Bologna, Italy, Rafael Bombelli (born c.1530) and Pietro Cataldi (1548-1626) must also be included in the history of continued fractions because they contributed examples of continued fractions for irrational numbers.[7] Bombelli expressed $\sqrt{13}$ as a repeating continued fraction. Cataldi did the same for $\sqrt{18}$. Besides these examples, however, neither mathematician investigated the properties of continued fractions.

Continued fractions became a field in its right through the work of John Wallis (1616-1703).[7][5] In his book *Arithemetica Infinitorium* (1655), he developed and presented the identity

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \cdots}$$

The first president of the Royal Society, Lord Brouncker (1620-1684), transformed this identity into

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\ddots}}}}$$

Though Brouncker did not dwell on the continued fraction, Wallis took the initiative and began the first steps towards generalizing continued fraction theory.

In his book *Opera Mathematica* (1695) Wallis laid some of the basic groundwork for continued fractions. He explained how to compute the n^{th} convergent and discovered some of the now familiar properties of convergents. It was also in this work that the term *continued fraction* was first used.

The Dutch mathematician and astronomer Christiaan Huygens (1629-1695) was the first to demonstrate a practical application of continued fractions.[7][6] He wrote a paper explaining how to use the convergents of a continued fraction to find the best rational approximations for gear ratios. These approximations enabled him to pick the gears with the correct number of teeth. His work was motivated in part by his desire to build a mechanical planetarium.

While the work of Wallis and Huygens began the work on continued fractions, the field of continued fractions began to flourish when Leonard Euler (1707-1783), Johan Heinrich Lambert (1728-1777), and Joseph Louis Lagrange (1736-1813) embraced the topic. Euler laid down much of the modern theory in his work *De Fractionlous Continious* (1737).[5] He showed that every rational can be expressed as a terminating simple continued fraction. He also provided an expression for e in continued fraction form. This identity is expressed below.

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}}}$$

He used this expression to show that e and e^2 are irrational. He also demonstrated how to go from a series to a continued fraction representation of the series, and conversely.

Lambert generalized Euler's work on e to show that both e^x and $\tan x$ are irrational if x is rational.[5] Lagrange used continued fractions to find the value of irrational roots.[5] He also proved that a real root of a quadratic irrational is a periodic continued fraction.

The nineteenth century can probably be described as the golden age of continued fractions. As Claude Brezinski writes in *History of Continued Fractions and Padé Approximations*, "the nineteenth century can be said to be the popular period for continued fractions."[2] It was a time in which "the subject was known to every mathematician."[2] As a result, there was an explosion of growth within this field. The theory concerning continued fractions was significantly developed, especially that concerning the convergents. Also studied were continued fractions with complex variables as terms. Some of the more prominent mathematicians to make contributions to this field include Jacobi, Perron, Hermite, Gauss, Cauchy, and Stieljes.[2] By the beginning of the 20th century, the discipline had greatly advanced from the initial work of Wallis.

Since the beginning of the 20th century continued fractions have made their appearances in other fields. To give an example of their versatility, a recent paper by Rob Corless examined the connection between continued fractions and chaos theory.[3] Continued fractions have also been utilized within computer algorithms for computing rational approximations to real numbers, as well as solving indeterminate equations.

This brief sketch into the past of continued fractions is intended to provide an overview of the development of this field. Though its initial development seems to have to taken a long time, once started, the field and its analysis grew rapidly. Even today, the fact that continued fractions are still being used signify that the field is still far from being exhausted.

3 Theory

In this section, I present some of the basic theorems that involve continued fractions. I have split this section into three subsections. The first discusses continued fractions and rational numbers. The second subsection deals with the convergents of a continued fraction. The final section is on continued fractions and irrational numbers.

3.1 Continued Fractions and Rational Numbers

We begin by proving a theorem about the connection between finite simple continued fractions and rationals. The proof has been developed from [7] and [6].

Theorem 1 *A number is rational if and only if it can expressed as a simple finite continued fraction*

PROOF: Let α be a rational number. Then $\alpha = \frac{p}{q}$ for some integers p and q, $q \neq 0$. Suppose also that α is in lowest terms, that is, p and q are relatively prime. To prove the statement, we make use of Euclid's Algorithm. By applying this algorithm, we can write

$$\begin{aligned}
p &= a_1 q + r_1, && 0 \le r_1 < q, \\
q &= a_2 r_1 + r_2, && 0 \le r_2 < r_1, \\
r_1 &= a_3 r_2 + r_3, && 0 \le r_3 < r_2, \\
&\vdots \\
r_{n-3} &= a_{n-1} r_{n-2} + r_{n-1}, && 0 \le r_{n-1} < r_{n-2}, \\
r_{n-2} &= a_n r_{n-1}.
\end{aligned}$$

The sequence $r_1, r_2, r_3, \ldots, r_k, \ldots$ forms a strictly decreasing sequence of non-negative integers that must become zero in a finite number of steps. Our notation is chosen so that $r_n = 0$ and $r_i \neq 0$ for $0 < i < n$.

The next step involves rearranging the algorithm in the following manner.

$$
\begin{aligned}
\frac{p}{q} &= a_1 + \frac{1}{\frac{q}{r_1}} \\
\frac{q}{r_1} &= a_2 + \frac{1}{\frac{r_1}{r_2}} \\
\frac{r_1}{r_2} &= a_3 + \frac{1}{\frac{r_2}{r_3}} \\
&\vdots \\
\frac{r_{n-2}}{r_{n-1}} &= a_{n-1} + \frac{1}{\frac{r_{n-1}}{r_n}} \\
\frac{r_{n-1}}{r_n} &= a_n
\end{aligned}
$$

Now, substituting each equation into the previous, we find that

$$
\alpha = \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}
$$

which is a finite simple continued fraction, as desired.

To show the converse, we prove by induction that if a simple continued fraction has n terms, it is rational. Let X represent the value of the continued fraction. We first check the base case $n = 1$. Then

$$X = a_1$$

But then X is clearly a rational, since a_1 is an integer.

We now prove the inductive case. Assume the theorem is true for all continued fractions having n terms. We now show that the theorem also holds for continued fractions with $n + 1$ terms. Let X be the value of a continued fraction that has $n+1$ terms. We wish to show that X is rational.

So, we have

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}}}}}.$$

Note, however, that we can rewrite this expression as

$$X = a_1 + \frac{1}{B},$$

where B is the continued fraction

$$B = a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}}}}}.$$

But B is a continued fraction with n terms, and by our induction hypothesis, it can be written as a rational $\frac{p}{q}$. This implies that

$$X = a_1 + \frac{1}{\frac{p}{q}}.$$

By applying some simple algebra, we arrive at the following equality,

$$X = \frac{a_1 p + q}{p}.$$

Since a_1, as well as p and q, is an integer, X must be a rational. Thus, the theorem is true for $n + 1$, and by induction, it must hold for all integers. □

It should be noted that the simple continued fraction expansion of a rational is not necessarily unique. For example, the simple continued fraction

$$[a_1; a_2, a_3, \ldots, a_{n-1}, a_n]$$

is equivalent to the simple continued fraction

$$[a_1; a_2.a_3, \ldots, a_{n-1}, (a_n - 1), 1]$$

since

$$a_n = (a_n - 1) + \frac{1}{1}$$

if $a_n > 1$. These two expressions, however, are the only possible two for any given rational. Of the two, the first is generally preferred.

3.2 The Convergents of Continued Fractions

One concept in the theory of continued fractions that cannot be glossed over is that of the convergents of a continued fraction. To ignore this topic is to ignore a central feature of continued fractions. Going back to our initial example of sequences and series in the introduction, the series

$$\sum_{i=1}^{N} a_i$$

is an approximation to the series

$$\sum_{i=1}^{\infty} a_i.$$

In fact, by definition

$$\lim_{N\to\infty} \sum_{i=1}^{N} a_i = \sum_{i=1}^{\infty} a_i.$$

The same thing is true with continued fractions. The continued fraction

$$C_k = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{}{a_{k-1} + \cfrac{1}{a_k}}}}$$

is an approximation to the simple continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}$$

In fact, it can be shown that if $\lim_{k\to\infty} C_k$ exists and is c, we say that c is the value of the infinite continued fraction. This is expressed as

$$\lim_{k\to\infty} C_k = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}.$$

The convergents of the continued fraction, as we will soon see, can be very useful. They give us the ability to solve indeterminate equations such as Diophantine and Pell's equations. In the following, I prove two important theorems about these convergents. The first theorem describes how we can calculate convergents through a recursive definition, while the second provides an important relationship between successive convergents. The proofs of [7] and [9] have been used as a source for these proofs.

Theorem 2 *Given a simple continued fraction with the terms,*

$$[a_1; a_2, a_3, \ldots, a_n, \ldots],$$

the numerator p_i and denominator q_i of the i^{th} convergent are defined for all $i \geq 0$ by the recursive definition

$$p_i = a_i p_{i-1} + p_{i-2}$$

$$q_i = a_i q_{i-1} + q_{i-2}$$

where $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$, and $q_0 = 0$. Note that in this cases, a_i can be any complex value.

What we are going to do in this proof is show that

$$\begin{aligned}
C_1 &= [a_1;] = \frac{p_1}{q_1} \\
C_2 &= [a_1; a_2] = \frac{p_2}{q_2} \\
C_3 &= [a_1; a_2, a_3] = \frac{p_3}{q_3} \\
&\vdots
\end{aligned}$$

If this is indeed true, it will give us a simple way to calculate the convergents of the continued fraction.

PROOF: We will prove this statement by using induction. Let the simple continued fraction

$$[a_1; a_2, a_3, \ldots, a_{n-1}, a_n, \ldots]$$

be given. The continued fraction can be infinite or finite. We need to first check the two base cases.

$$\begin{aligned} C_1 &= a_1 = \frac{a_1}{1} = \frac{a_1 1 + 0}{a_1 0 + 1} = \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}} = \frac{p_1}{q_1} \\ C_2 &= a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2} = \frac{a_2 a_1 + 1}{a_2 1 + 0} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2} \end{aligned}$$

Both cases agree with the definition. We now assume that the statement is true for the positive integer k. We wish to show that the statement is true for $k+1$.

$$C_{k+1} = [a_1; a_2, a_3, \ldots, a_{k-1}, a_k, a_{k+1}].$$

We can rewrite this fraction in the following manner

$$C_{k+1} = [a_1; a_2, a_3, \ldots, \left(a_k + \frac{1}{a_{k+1}}\right)].$$

The continued fraction now has k terms, where each term is a complex value, and by hypothesis

$$\begin{aligned} C_{k+1} &= \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} \\ &= \frac{(a_k a_{k+1} + 1)p_{k-1} + a_{k+1} p_{k-2}}{(a_k a_{k+1} + 1)q_{k-1} + a_{k+1} q_{k-2}} \\ &= \frac{a_k a_{k+1} p_{k-1} + p_{k-1} + a_{k+1} p_{k-2}}{a_k a_{k+1} q_{k-1} + q_{k-1} + a_{k+1} q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} \\ &= \frac{p_{k+1}}{q_{k+1}} \end{aligned}$$

The second last step made use of the induction hypothesis for the substitution. The theorem is thus true for $k+1$, and by induction, must hold for all integers. $\square$

To demonstrate how this algorithm works, I have provided a simple example. Consider the following simple finite continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}.$$

Then, by applying the previous theorem, we see that

$$\begin{aligned} p_1 &= a_1 p_0 + p_{-1} = 1(1) + 0 = 1 \\ p_2 &= a_2 p_1 + p_0 = 2(1) + 1 = 3 \\ p_3 &= a_3 p_2 + p_1 = 3(3) + 1 = 10 \\ p_4 &= a_4 p_3 + p_2 = 4(10) + 3 = 43 \end{aligned}$$

and

$$\begin{aligned} q_1 &= a_1 q_0 + q_{-1} = 1(0) + 1 = 1 \\ q_2 &= a_2 q_1 + q_0 = 2(1) + 0 = 2 \\ q_3 &= a_3 q_2 + q_1 = 3(2) + 1 = 7 \\ q_4 &= a_4 q_3 + q_2 = 4(7) + 2 = 30 \end{aligned}$$

Using these results, we can easily compute the convergerts of the continued fraction. They are $C_1 = 1$, $C_2 = \frac{3}{2}$, $C_3 = \frac{10}{7}$, and $C_4 = \frac{43}{30}$.

One other theorem that involves convergents demonstrates a relationship between successive convergents, or more specifically, between their numerators and denominators.

Theorem 3 *If p_k and q_k are defined as in the above theorem, then*

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^k$$

for all $i \leq 0$.

PROOF: To prove this, we will once again do a proof by induction We first check the two base cases, that is, for $i = 1$ and $i = 2$.

$$p_0 q_{-1} - p_{-1} q_0 = 1(1) - 0(0) = 1 = (-1)^0$$

$$\begin{aligned} p_1 q_0 - p_0 q_1 &= (a_1 p_0 + p_{-1})(0) - 1(a_1 q_0 + q_1) \\ &= 0 - 1(0+1) \\ &= -1 \\ &= (-1)^1 \end{aligned}$$

Now we show that the above theorem holds for all k. Assume that the theorem is true for the positive integer k. We want to show that the statement is true for $k+1$.

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= a_{k+1} p_k q_k + p_{k-1} q_k - a_{k+1} p_k q_k - p_k q_{k-1} \\ &= p_{k-1} q_k - p_k q_{k-1} \\ &= -(p_k q_{k-1} - p_{k-1} q_k) \\ &= -(-1)^k \\ &= (-1)^{k+1} \end{aligned}$$

Since the statement is true for $k+1$, by induction, the theorem holds for all integers k. □

This completes a brief introduction to convergents. We will, however, return to the subject when we discuss applications of continued fractions.

3.3 Continued Fractions and Irrational Numbers

Given any irrational number α, we can express α as a continued fraction by using the following recursive definition:

$$\begin{aligned} a_i &= \lfloor \alpha_i \rfloor \\ \alpha_{i+1} &= \frac{1}{\alpha_i - a_i} \end{aligned}$$

where $\alpha_1 = \alpha$ and the function $\lfloor \gamma \rfloor$ denotes the greatest integer less than or equal to γ. This algorithm is attributed to Euler.

For example, consider the irrational number $\pi = 3.1415926535.....$ We first let $\alpha_1 = \pi$. Then

$$a_1 = \lfloor \pi \rfloor = 3$$

$$\begin{aligned}
\alpha_2 &= \frac{1}{\pi - 3} = 7.0625133... \\
a_2 &= \lfloor 7.0625133... \rfloor = 7 \\
\alpha_3 &= \frac{1}{7.0625133... - 7} = 15.99659... \\
a_3 &= \lfloor 15.99659... \rfloor = 15 \\
&\vdots
\end{aligned}$$

Thus, the continued fraction for π is

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{\ddots}}}$$

Note, that from this continued fraction, we can compute its convergents. These convergents provide us with rational approximations to π. The first three are:

$$\begin{aligned}
C_1 &= 3 \\
C_2 &= \frac{22}{7} \\
C_3 &= \frac{333}{106}
\end{aligned}$$

The continued fraction expansion of any irrational number α, it should be noted, has an infinite number of terms. In the next theorem, we formalize this statement.

Theorem 4 *If α is an irrational number, then its simple continued fraction expansion is infinite.*

PROOF: Let α be an irrational number, and suppose that its simple continued fraction is finite. Then the simple continued fraction has n terms where n is a postive integer. But by Theorem 1, the value of any continued fraction with a finite number of terms must be rational. Hence the continued fraction is equivalent to a rational, and thus, it cannot be equivalent to α. This provides us with the necessary contradiction.$\square$

We can now claim that we can convert any real number α into a continued fraction. The converse is also true. In the next theorem, we state this idea more formally without proof.

Theorem 5 (Euler's Theorem) *Every simple continued fraction converges to a unique real number. Conversely, for any real number there is a simple continued fraction which converges to it.*

While the set of irrational numbers can be broken down up into various subsets, ie., algebraic and transcendental, the next theorem will deal with a specific subset of these numbers, namely quadratic irrationals. Before I can prove a theorem on the connection between quadratic irrationals and continued fractions, it is necessary to first provide two definitions, as well as a lemma, that have been modeled on [9].

Definition A *quadratic irrational* refers to all numbers of the form

$$\frac{A+\sqrt{B}}{C}$$

where A, B, and C, are integers (B must also be positive and non-square and C must be non-zero). They are called quadratic irrationals since they are irrational roots of quadratic equations, specifically of

$$C^2x^2 - 2ACx + (A^2 - B) = 0.$$

Definition The infinite simple continued fraction $[a_1; a_2, a_3, \ldots]$ is said to be *periodic* if there are postive integers N and k such that for all $n \geq N$, $a_n = a_{n+k}$. We represent this continued fraction

$$[a_1; a_2, a_3, \ldots, a_{N-1}, a_N, a_{N+1}, \ldots, a_{N+k-1}, a_N, a_{N+1}, \ldots]$$

by the more effecient notation

$$[a_1; a_2, a_3, \ldots, a_{N-1}, \overline{a_N, a_{N+1}, \ldots, a_{N+k-1}}].$$

Lemma 1 *If α is a quadratic irrational, and $r, s, t,$ and u are integers, then $\frac{r\alpha+s}{t\alpha+u}$ is either a quadratic irrational, or rational.*

PROOF: Let α be a quadratic irrational. Then α can be rewritten as $\frac{a+\sqrt{b}}{c}$, where a, b, and c are integers, b is a positive non-sqare integer, and $c \neq 0$. Thus

$$\frac{r\alpha+s}{t\alpha+u} = \frac{r(\frac{a+\sqrt{b}}{c})+s}{t(\frac{a+\sqrt{b}}{c})+u}$$

$$
\begin{aligned}
&= \frac{ra + r\sqrt{b} + sc}{ta + t\sqrt{b} + uc} \\
&= \frac{(ra + sc) + r\sqrt{b}}{(ta + uc) + t\sqrt{b}} \\
&= \left(\frac{(ra + sc) + r\sqrt{b}}{(ta + uc) + t\sqrt{b}}\right) \cdot \left(\frac{(ta + uc) - t\sqrt{b}}{(ta + uc) - t\sqrt{b}}\right) \\
&= \frac{(ra + sc)(ta + uc) + [r(ta + uc) - t(ra + sc)]\sqrt{b}}{(ta + uc)^2 - t^2 b}
\end{aligned}
$$

We now see that $\frac{r\alpha+s}{t\alpha+u}$ is a quadratic irrational since it is of the form $\frac{P+\sqrt{Q}}{R}$. It is rational if the coeffecient of $\sqrt{b}$ in the last equation above is zero. This will occur if $ru = ts$. □

With these definitions and this lemma, we can now prove the following theorem about continued fractions and irrational numbers.

Theorem 6 *If the infinite simple continued fraction of an irrational number is periodic, then the irrational number is a quadratic irrational.*

PROOF: Let α be an irrational number whose continued fraction is periodic. Then $\alpha = [a_1; a_2, a_3, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+k-1}}]$. Let β represent the periodic portion of the continued fraction. Thus,

$$\beta = [\overline{a_{N+1}, \ldots, a_{N+k-1}}].$$

We can rewrite this as

$$\beta = [a_{N+1}, \ldots, a_{N+k-1}, \beta].$$

By applying our knowledge of convergents, we see that

$$\frac{p_k}{q_k} = \beta = \frac{\beta p_{k-1} + p_{k-2}}{\beta q_{k-1} + q_{k-2}}.$$

In this case, $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{p_{k-2}}{q_{k-2}}$ are the $(k-1)^{th}$ and $(k-2)^{th}$ convergents of $[a_{N+1}, \ldots, a_{N+k-1}]$ respectively. Solving for β in the above equation, we find that

$$\beta^2 q_{k-1} + (q_{k-2} - p_{k-1})\beta - p_{k-2} = 0.$$

We see that β is a quadratic irrational since it is the root of a quadratic equation. Note that β cannot be a rational root since the continued fraction of β is infinite.

We can now rewrite α as

$$\alpha = [a_1; a_2, a_3, \ldots, a_N, \beta]$$

Again, we apply the definition of convergents to find that

$$\alpha = \frac{p_{N+1}}{q_{N+1}} = \frac{\beta p_N + p_{N-1}}{\beta q_N + q_{N-1}}$$

where $\frac{p_N}{q_N}$ and $\frac{p_{N-1}}{q_{N-1}}$ are the N^{th} and $(N-1)^{th}$ convergents of $[a_1; a_2, a_3, \ldots, a_N]$. By our lemma, we see that α must be either a quadratic irrational or a rational. However, α cannot be rational since its continued fraction is infinite. Thus, α is a quadratic irrational, and hence, all periodic continued fractions are equal to some quadratic irrational. $\square$

We provide the following example to demonstrate this theorem. Consider the following periodic continued fraction where x is the value of the continued fraction.

$$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}.$$

By the theorem just proved, x should be a quadratic irrational. We can rewrite the continued fraction as

$$x = 2 + \frac{1}{x}.$$

We then rearrange the equation to get

$$x^2 - 2x - 1 = 0.$$

Solving for x, we find that

$$x = 1 \pm \sqrt{2}.$$

However, since x is positive, we let x equal the positive root, $1 + \sqrt{2}$, which is indeed a quadratic irrational.

The converse of this theorem is true as well. A proof is omitted from this discussion since it requires a number of lemmas, an explanation of reduced quadratic irrationals, as well as a description of a recursive algorithm for computing the terms of a continued fraction. This theorem and the one we just proved are combined into one theorem named after Lagrange who first proved the converse of Theorem 6. The statement of the theorem can be found below.

Theorem 7 (Lagrange's Theorem) *The infinite simple continued fraction of an irrational number is periodic if and only if this number is a quadratic irrational.*

A proof of the converse can be found in [9].

4 Applications

Continued fractions, like many other fields of mathematics, also has an applied aspect. Recall from the historical overview of the field that the Indian mathematician Āryabhata used continued fractions to solve indeterminate equations. The Dutch astronomer Huygens used continued fractions to find which gears should be used to create a desired gear ratio. In this section, I explain these two applications in greater detail.

4.1 Continued Fractions and Indeterminate Equations

The equation $4x + 7y = 13$ is said to be an indeterminate equation because any solution for y depends on the value choosen for x. The solutions cannot be determined solely from the given information, hence the use of the term indeterminate. In number theory we are interested in finding all integer solutions to such equations.

Two indeterminate equations that have received much attention over the centuries have the following forms:

$$ax + by = c \tag{1}$$

$$x^2 - Py^2 = 1 \tag{2}$$

where a, b, and c are integers and P is any positive integer that is not a perfect square. Equations that have the form of (1) are generally referred to as *linear Diophantine equations* after the 3rd century A.D. Greek mathematician Diophantus.[7] The equations that have the form of (2) are called *Pell's equations*, named after John Pell (1611-1685).[7]

Though continued fractions can be used to solve both equations, I will concentrate on how to use continued fractions to find all the interger solutions to a linear Diophantine equation (See [7] for more). It should be noted that the program that I have written, which will be described later, contains functions that enable one to solve both Pell's equations and linear Diophantine equations. The user must input the non-square value for P, or the integer values of a,b, and c. The program will then compute all integer solutions using an algorithm that uses continued fractions. In order to solve Pell's equations using continued fractions, it is first necessary to define a recursive definition for computing the continued fraction of a quadratic irrational. For this reason, we skip over this topic and discuss only linear Diophantine equations. See [9] and [6] for more on the relationship between continued fractions and Pell's equations.

Solving linear Diophantine equations can be broken down into the simpler problem of solving for the equation $ax + by = 1$. Once we have found one pair of integer soulutions x_0, y_0), we can solve the specific problem of $ax + by = c$ by multiplying each solution by c since

$$\begin{aligned} a(cx_0) + b(cy_0) &= c(ax_0 + by_0) \\ &= c(1) \\ &= c \end{aligned}$$

It should also be pointed out that once we have found one solution to the equation, we can use it to find all other integer solutions. Suppose (x_0, y_0) is a solution to equation (1). Then $((x_0 - bn), (y_0 + an))$, where n is any integer, is also a solution. This can be seen quite easily as demonstrated below.

$$\begin{aligned} a(x_0 - bn) + b(y_0 + an) &= ax_0 - abn + by_0 + abn \\ &= ax_0 + by_0 \\ &= c \end{aligned}$$

Before I demonstrate how to use continued fractions to solve linear Diophantine equations, I must mention one restriction that should be placed on a and b. The values of a and b should be relatively prime. If they are not, then a and b might have a common factor that c does not. If this is the case, there are no solutions since one side of the equation is divisible by this factor, while the other is not. For example, there are no integer solutions to the equation $4x + 2y = 7$ since the left side is divisible by 2, but the right side is not.

By imposing this condition on a and b, that is, $\gcd(a,b) = 1$, we know we can find *all* solutions to the Diaphantine equation. Suppose that (x_0, y_0)

and (x_1, y_1) are solutions to the same Diaphantine equation. We want to show that

$$(x_1, y_1) = (x_0, y_0) + n(-b, a)$$

for some n. Since

$$\begin{aligned} ax_0 + by_0 &= c = ax_1 + by_1 \\ ax_0 - ax_1 &= by_1 - by_0 \\ a(x_0 - x_1) &= b(y_1 - y_0) \end{aligned}$$

Since gcd(a,b) $= 1$, b must divide $(x_0 - x_1)$ and a must divide $(y_1 - y_0)$. So,

$$\begin{aligned} (x_0 - x_1) &= bn \\ (y_1 - y_0) &= am \end{aligned}$$

for some integers n and m. But $n = m$, and this gives us our desired n. So, all solutions to the Diaphantine equation can be found by this method.

We are now in a position to explain how to use continued fractions to solve equations of the form $ax + by = 1$. To solve such equations, we must first form the continued fraction of $\frac{a}{b}$. Let $[a_1; a_2, \ldots, a_n]$ be this continued fraction. (Since $\frac{a}{b}$ is a rational, its simple continued fraction will have a finite number number of terms by Theorem 1.) Recall the convergent relationship that we proved earlier, that is,

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^k.$$

We also know that the n^{th} convergent, $C_n = \frac{p_n}{q_n}$, of $[a_1; a_2, \ldots, a_n]$ must be equal to $\frac{a}{b}$, So,

$$aq_{n-1} + b(-p_{n-1}) = (-1)^n.$$

From this equation, we see that if the number of terms of the continued fraction is even, then a solution to the equation $ax + by = 1$ is q_{n-1} and $(-p_{n-1})$, where $\frac{p_{n-1}}{q_{n-1}}$ is the $(n-1)^{th}$ convergent.

However, if the number of terms is odd, we have

$$aq_{n-1} + b(-p_{n-1}) = -1$$

By multipling through by -1, we find that

$$a(-q_{n-1}) + bp_{n-1} = 1$$

thus giving us a desired solution. With these solutions, we can find the solution to the equation $ax + by = c$ as described above.

I provide here a brief example. Suppose we wished to find all integer solutions to the equation $7x + 19y = 23$. Our first step is to find the continued fraction of $\frac{7}{19}$. When we do, we find that

$$\frac{7}{19} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}$$

Next, we caclulate the convergents of the above fraction. We see that $C_1 = \frac{0}{1}$, $C_2 = \frac{1}{2}$, $C_3 = \frac{1}{3}$, $C_4 = \frac{3}{8}$, and $C_5 = \frac{7}{19}$. Since there are an odd number of convergents, our solution to the problem $7x + 19y = 1$ is $x = (-q_4) = -8$ and $y = p_4 = 3$. We first check to see if this true:

$$\begin{aligned} 7 \cdot (-8) + 19 \cdot (3) &= -56 + 57 \\ &= 1 \end{aligned}$$

We can now find the solution to our specfic problem by multiplying each solution by 23. We find that one solution to the equation is $7x + 19y = 23$ is $(-184, 69)$. From this solution, we can find the set of all integer solutions. The set of all solutions is $\{(-184 - 19n), (69 + 7n)|n \in Z\}$.

An interesting off shoot of this work is that it also enables us to solve equations of the form

$$ax \equiv b \pmod{m}.$$

This congruence problem can be solved by using some of the methods we have just discussed. To do this, we note that solving this equation is equivalent to solving the equation

$$ax + my = b$$

which is a linear Diophantine equation. If (x_0, y_0) are solutions to this equation, then

$$ax_0 + my_0 = b$$

But then x_0 is a solution to the above congruence since

$$\begin{aligned} ax_0 \pmod{m} &\equiv ax_0 + my_0 \pmod{m} \\ &\equiv b \pmod{m} \end{aligned}$$

Hence, our method to solve linear Diophantine equations can be extended to solve other problems, as in this case, were we use it to solve congruence problems.

In general, linear Diophantine equations can be solved by taking advantage of the relationship we found between consecutive convergents. By using these results, we can find the entire set of solutions.

4.2 Continued Fractions and Gear Ratios

One of the first applied uses of continued fractions was to determine which gears to use in creating a desired gear ratio. This application, which was first pioneered by the Dutch mathematican and astronomer Christiaan Huygens, is the focus of this section. We shall see how to utilize continued fractions for this purpose. In addition to this, we will prove a theorem that justifies this choice. The major source for this discussion is [6].

The term *gear ratio* is used to refer to the number of rotations a gear must make to cause its companion gear to revolve a different number of rotations. For example, if two gears, say A and B, are joined together and have a gear ratio of 3:2, this means that gear A must revolve three times to cause two rotations of gear B. (This is equivalent to saying that gear B must rotate twice to cause three revolutions of gear A.) To create a gear ratio, say X:Y, one merely has to use a gear with X number of teeth and a gear with Y teeth. For example, to create a gear ratio of 37:51, one has to use a gear with 37 teeth combined with a gear of 51 teeth.

However, determining which gears to use is not always this easy. Two difficulties immediately arise. The first problem is a result of the impractibility of producing gears with any given number of teeth. Generally, the number of teeth on gear vary between twenty and a hundred. Any less causes the gears to mesh incorrectly; any more is difficult to make. Thus, a gear ratio of 101:201 cannot be made by using gears with teeth that have 101 and 201 teeth respectively.

The second difficulty arises when one wishes to institute a gear ratio that is irrational. For example, suppose one needs a ratio of $\sqrt{101} : 45$. In this case, it is meaningless to pick a gear with $\sqrt{101}$ teeth. To get around this problem and the previous one, we need to find rational approximants that have both their numerator and denominator between twenty and a hundred.

To get around both of these problems, we can make use of continued fractions, or more specifically, the convergents of a continued fraction to find rational approximants. The first step in finding these approximants is to compute the continued fraction of the ratio. In the case of an irrational number, computing the first dozen or so terms should be sufficient. If not, we can always compute more. The next step involves calculating the convergents of the continued fraction. Once we have done this, we search our set of convergents for those with both numerator and denominator in our our desired range, that is, between twenty and a hundred. If there are more than one, we pick the convergent with the greatest subscript. This convergent is the one we will use to approximant the desired ratio.

Before I justify some of the steps in this algorithm, I will demonstrate how it is used. Suppose the desired ratio we need is 7111:10000. Obviously we cannot use gears of 7111 and 10000 teeth, so we need to find a rational approximant that has both its numerator and denominator between twenty and hundred. We calculuate the contined fraction and find that

$$\frac{7111}{10000} = [0; 1, 2, 2, 5, 1, 43, 1, 1, 2].$$

Next, we calculate the convergents of the continued fraction. The values are listed below.

$$C_1 = \frac{0}{1}, \qquad C_6 = \frac{32}{45},$$

$$C_2 = \frac{1}{1}, \qquad C_7 = \frac{1403}{1973},$$

$$C_3 = \frac{2}{3}, \qquad C_8 = \frac{1435}{2018},$$

$$C_4 = \frac{5}{7}, \qquad C_9 = \frac{2838}{3991},$$

$$C_5 = \frac{27}{38}, \qquad C_{10} = \frac{7111}{10000}.$$

Only C_5 and C_6 have both their numerator and denominator between twenty and a hundred. Since C_6 has the greater subscript, we pick $\frac{32}{45}$ as our rational approximant for $\frac{7111}{10000}$. We can now create our gear ratio by using gears with teeth of 32 and 45 respectively. Notice that $\frac{32}{45}$ overestimates $\frac{7111}{10000}$ by only a small value. In fact,

$$\left|\frac{7111}{10000} - \frac{32}{45}\right| = 0.0000\overline{1}$$

The approximation is quite close to the desired ratio.

Note that I have not explained why this works, and why we pick the convergent with the greatest subscript. In the following, I try to fill in these gaps.

Part of the reason this method works is based on the fact that the convergents converge to the value of the continued fraction. Suppose α is this value. Then it can be shown that the odd convergents is an *increasing* sequence that converges to α, that is

$$C_1 < C_3 < C_5 < \ldots \leq \alpha.$$

The value of α may equal one of the convergents if α is a rational whose continued fraction has an odd number of terms. The even convergents form a *decreasing* sequence that converges to α;

$$C_2 > C_4 > C_6 > \ldots \geq \alpha.$$

Again, α is in the sequence only if α is a rational whose continued fraction has an even number of terms. If α is irrational, the sequence approaches the value of α.

Rather than providing a proof for these claims, I will direct your attention to [7] and [9]. I will instead demonstrate why we chose the convergent with the greatest subscript. It is not readily apparent why we do this. We will see that each convergent is a better approximant to the continued fraction than the previous. This proof has been modeled upon the one found in [9].

Theorem 8 *Each convergent is nearer to the value of a simple continued fraction than is the preceding convergent.*

PROOF: We must consider two cases, when the simple continued fraction is finite, and when it is infinite. We will first show that the statement is true if the simple continued fraction is finite.

Let x be a rational number whose continued fraction is given by

$$[a_1; a_2, a_3, \ldots, a_n].$$

We now rewrite x as

$$x = [a_1; a_2, a_3, \ldots, a_k, x_{k+1}]$$

where

$$x_{k+1} = [a_{k+1}, \ldots, a_n].$$

Since x is rational, its continued fraction has n terms for some integer n We impose the condition on k such that $k \leq n-1$. If k were larger, then our definiton of x_{k+1} would be meaningless.

Using our definiton of convergents,

$$x = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}}.$$

Rearranging, we find that

$$x(x_{k+1}q_k + q_{k-1}) = x_{k+1}p_k + p_{k-1}.$$

For $k \geq 2$, we have

$$\begin{aligned} x_{k+1}(xq_k - p_k) &= -(xq_{k-1} - p_{k-1}) \\ &= -q_{k-1} \cdot \left(x - \frac{p_{k-1}}{q_{k-1}}\right) \end{aligned}$$

Next, we divide through by $x_{k+1}q_k$ to get

$$\left(x - \frac{p_k}{q_k}\right) = \left(-\frac{q_{k-1}}{x_{k+1}q_k}\right) \cdot \left(x - \frac{p_{k-1}}{q_{k-1}}\right).$$

We know that if $a = b \cdot c$, then $|a| = |b| \cdot |c|$. So,

$$\left|x - \frac{p_k}{q_k}\right| = \left|\frac{q_{k-1}}{x_{k+1}q_k}\right| \cdot \left|x - \frac{p_{k-1}}{q_{k-1}}\right|. \tag{3}$$

Since $2 \leq k \leq n+1$, we know that $x_{k+1} > 1$. If $k < n+1$, the $x_{k+1} = [a_{k+1}, \ldots, a_n]$. Since each term is positive, then $x_{k+1} > a_{k+1} \geq 1$. If $k = n-1$, then $x_{k+1} = [a_n]$. We can assume that $a_n > 1$ because if it is equal to one, we can rewrite the continued fraction for x as

$$[a_1; a_2, a_3, \ldots, (a_{n-1}+1)].$$

Similarly, we can show that for $k \geq 2$, $q_k > q_{k-1} > 0$. Recall that $q_i = a_i q_{i-1} + q_{i-2}$, where $q_0 = 0$, and $q_{-1} = 1$. When we calculate the first couple terms, we see that $q_1 = 1, q_2 = a_2, q_3 = a_3a_2 + 1, \ldots$ Since $a_i \geq 1$ for all $i \geq 2$, then the sequence $q_1, q_2, q_3, \ldots$ must be strictly increasing.

Hence,

$$0 < \frac{q_{n-1}}{x_{n+1}q_n} < 1,$$

or

$$0 < \left|\frac{q_{n-1}}{x_{n+1}q_n}\right| < 1.$$

By applying the above to (3), we see that

$$\left|x - \frac{p_k}{q_k}\right| < \left|x - \frac{p_{k-1}}{q_{k-1}}\right|$$

for $n \geq 2$, or by rewritten in terms of convergents, we arrive at

$$|x - C_k| < |x - C_{k-1}|$$

for $n \geq 2$. This demonstrates that C_k is closer to x then C_{k-1}.

The proof is similar to show that the theorem is true for infinite simple continued fractions. The difference is that we assume that

$$x = [a_1; a_2, a_3, \ldots, x_{k+1}]$$

where

$$x_{k+1} = [a_{k+1}, a_{k+2}, \ldots].$$

The proof follows just as before except this time we do not impose a restriction on the size of k. □

Before I conclude on this topic, there are some difficulties I should mention with this algorithm. This algorithm, or process, does not always work. There may be cases where none of the convergents have both their numerator and denominator between twenty and a hundred. For example, the first few convergents for π are $C_1 = \frac{3}{1}, C_2 = \frac{22}{7}, C_3 = \frac{333}{106}$, and $C_4 = \frac{355}{113}$. Note that none of the convergents have the property for which we are looking. One way around this problem is to find a fraction that is equivalent to one of these convergents that is within the desired range. For example, we can multiply both the numerator and denominator of C_2 by four to give us a rational approximation of $\frac{88}{28}$. This is a rational we can use. But even then, not *all* solutions can be found. There exist some gear ratios that cannot be made by joining two gears together. Methods other than continued fractions, which will not be discussed here, must be employeed.

Continued fractions, though they do not provide all solutions, enable us to find which gears to use to create a desired gear ratio. The algorithm takes advantage of the convergents of the continued fraction to find rational approximants for the needed gear ratio.

5 Putting Continued Fractions On-line

Up to this point, my focus has been on the historical past of continued fractions. I first delved into the history of continued fractions. I followed this by explaining and proving some classical theorems about continued fractions. I also discussed past applications of this discipline. At this point, however, I would like to shift the focus away from past developments and discuss the other part of my independent study, that is, my attempt at combining continued fractions and the World Wide Web.

The World Wide Web (WWW) is a facet of the Internet that has experienced an explosion of growth and use. In a few short years, this medium has become the premiere way to navigate the internet. My first introduction to the WWW, or web as it is commonly called, came from one of the mathematics department's colloquia. I was further immersed into the web when I received an undergraduate scholarship to work at the Centre for Experimental and Constructive Mathematics in Vancouver, BC, during the summer of 1995. It was here that I learned to create web pages and programs that web users could interact with. This job would later prove to be invaluable experience for this project.

On the web, one can find information on almost every topic. When I began this project, I searched the web for information on continued fractions. However, most of my searches resulted in finding only mathematical papers that mentioned or used continued fractions. I could not find a site that could provide an elementary introduction to the topic. Finding this void provided some of the initial inspiration for creating a web site devoted to continued fractions. This project would also enable me to indulge in one of my other interests, that is, computers and programming. I envisioned this web site as a thorough introduction to continued fractions. Not only would it include some historical background and some basic theorems, I also wanted to make interactive programs to give the general user a flavor of this field and its possiblities.

Rather than jumping into the development of the web site, I decided to first create a library of functions that used and calculated continued fractions. Once I had created this library, I could then use it for the interactive feature for my web site. An advantage of creating the library first was the fact that it would in some sense force me to truly understand continued fractions. Only after understanding continued fractions would I be able to code the algorithms that utilized them.

For a programming language, I decided to write in C++ since I already had some working knowledge of this language. Most of the work was done on the school's Sparc workstations that use the Sun operating system. Over the course of a semester, I programmed various functions that eventually ended up in the library. For example, I designed a function that would receive as input a rational value and output the simple continued fraction expression for that rational. I did the same for quadratic irrationals. I also developed a method to evalulate continued fractions. A function that computed the convergents of a continued fraction was included as well. As noted before, the library contains functions that enable one to compute the solutions of linear Diophantine and Pell's equations.

I should note that there exist other libraries that deal with continued fractions. For example, the symbolic mathematics programs, Maple V and Mathematica, include functions that do much the same thing as my library. (Actually, from what I could find, Mathematica cannot evaluate a non-terminating, periodic fraction as mine can.) I mention this to point out that my work is not unique and that it can be obtained via other sources. However, part of the challenge of this work was to see if I could do it myself.

In programming the library, I encountered some difficulty in implementing algorithms that used continued fractions. Presently, I have run into two major problems that, if time permits, I will try to correct. The first bug derives not from my algorithm, but rather, from the way integer values are stored on the computer. Any integer value larger than $2^{31} - 1$ cannot be accurately stored since not enough memory has been allocated for it. When such a value is entered, it is stored as an incorrect value. This prevents the program from running correctly since it does not have the correct value to use. Though this is not a serious problem, it can sometimes be annoying and prevents the program from being as stable as I wanted it to be. One possible solution for this problem is to declare all the integer values with the C++ `long int` data type. This should greatly increase the size of the integer that can be stored.

Another snag that I have discovered in my project is that I cannot always rely on my use of static arrays, that is, arrays that do not have the ability to change their storage capacity. Though I have set most of these arrays to a large size, some specific examples have resulted in the dreaded `segmentation fault` error. To correct this, I will need to rewrite a portion of the program and use dynamic arrays. These arrays, though more tricky to implement, should correct the problem.

Overall, the functions in the library seem to work as desired. In most simple cases it gives the correct response. The errors only show up in a small percentage of the inputs. If time permits, I plan to fix these errors in order to make the functions as robust as possible. For those of you interested, I have included the code for my library as an appendix to this paper.

Once I had completed the library of functions, I was ready to begin work on the actual web site. One of the first things that I needed to do was to determine the intended audience of this web site. This decision would greatly effect the content of the web site. For example, if I wanted to explain continued fractions to the average person, I would have to greatly limit the mathematical content. In the end, I decided to make this web site understandable to those at my own level, that is, undergraduates with an interest or aptittude for mathematics. I opted for this route because I figured it was this subset of web users that would probably want or need an introduction to continued fractions. This group would be the ones that would be able to understand and use my site. As a result, much of the material and content of the web site assumes at least some college mathematics.

The other thing that I needed to decide was how to lay out the web pages. I ended up splitting the web site into five main sections: Introduction, History, Theory, Applications, and Bibliography and Sources. The first section, Introduction, is what one would expect. It provides a brief overview of the site, as well as a motivation for its existence.

The next two divisions of the web site contain some of the material already presented in this paper. The History division provides a sketch of past developments of continued fractions. The Theory section provides a number of definitions, as well as a few proofs on continued fractions that are found in this paper. One major difference between this paper and the on-line web site was my ability to use hypertext. I used hypertext around difficult words, names, or bibliographical sources to allow the user to "jump" to related information at the click of a button. This ability is one of the advantages of writing documents using a hypertext language such as HTML (HyperText Markup Language).

The next section, Applications, is one that greatly differs from the section on applications in this paper. While in this paper I merely described how to use continued fractions, on the web site I provide the user with the ability to actually perform some of these applications. Those accessing this site can change a rational of their choice into a continued fraction, or choose to find the value of a continued fraction after providing its terms. It can also calculate every convergent of a given continued fraction. Users can also use this web site to find solutions to Pell's equations and Diophantine equations.

In order to provide this interaction, I needed to do a number of things.

First, I had to write the programs that used the library. These programs had to be written in C++ in order to take advantage of the library. Secondly , I had to create small programs, or scripts, written in Perl (Practical Extraction and Report Language), that would receive the input from the user. The Perl script had to verify that the users input was correct (that it wasn't a real value or a character) before passing the input along to the correct program. In a sense, the Perl scripts coordinate the action between the user and the actual program that executes the function. And thirdly, it was also necessary to create the web pages correctly to receive the input in the first place. The coordination between these three pieces, the web page, the Perl scripts, and the C++ programs provided some of the greater difficulty in this project.

The final division of the web site is a section devoted to a list of resources and a bibliography. It also contains a list of sites on the World Wide Web that deal with continued fraction. In order to find these sites, I used the presently available web searchers, such as Yahoo, Lycos, and Webcrawler. I then found it necessary to check each result to determine if the site contained some substance on the topic of continued fractions as opposed to a trivial reference.

One facet of this project that I should also mention is the effort to make the web site aesthetically pleasing. From my past experience "surfing" the net, I have come across a number of sites whose appearance have made it difficult to use. I spent some time making sure that the site was well laid out, as well as easy to use and read. Though this facet may have no direct connection to continued fractions, I thought it would be prudent to dedicate some time to this area. I did not want prospective users to be turned off from using this site just because of a poor choice of a background.

Presently, I have not publicly announced the completion of my web site on continued fractions. I thought it would be appropriate to first receive some local feedback before I make this step. When I have received some initial criticism and instituted any changes warranted by any suggestions, I will publicize my site.

With the lack of any feedback, it is hard to judge whether I have fulfilled my goal. What I think may be well laid out and explained could be disorder and incoherent to someone using my site. Hopefully, however, this site will received and used as I envisioned it, that is, as an introduction to the world of continued fractions. For those of you who wish to check out this web site, the URL (Uniform Resource Locator) of the site is: *http://www.calvin.edu/~avtuyl52/confrac/*

6 Conclusion

If I were asked to describe my independent study in a single word, the word I would probably settle upon would be *experiment.* I would pick this word because this is what this project has been for me. This independent study has been an experiment in doing mathematics on my own. Almost every aspect of this project involved trying something new, experimenting with its possiblities. For example, this project is the first time that I have had to do mathematics on my own. This included working out proofs by myself, (albeit, with some help from advisor), as well as learning to do independent research. One of my weekly assignments was to present my work to my advisor via a blackboard in his office. This gave me the opportunity to experiment with this medium. This project has also been my first experiment in writing a mathematics paper. Integrating continued fractions and the World Wide Web was another experiment that I tackled. I even experimented with using LaTeXto create this paper. Before this project is finished, I will also have tried my hand at teaching what I have learned at Calvin's weekly mathematics colloquim. And of course, I learned more about continued fractions, a field of mathematics that seems to have been relatively ignored over the last couple of years. It is my hope, that through this paper, and especially through the web site, that others will inspired to *experiment* with the field of continued fractions as well.

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Appendix

In the pages that follow, one will find the C++ code for the library of functions that I wrote as part of this project. There are two parts. The first part is the header file for the library which contains the description of the various functions. The second part is the code that actually carries out and performs these functions.

Appendix A - The Header File

Appendix B - Source Code of the Library