Some results on fat points whose support is a complete intersection minus a point

Elena Guardo and Adam Van Tuyl

Abstract. This paper is an investigation of some of the numerical invariants associated to a set of fat points $Z \subseteq \mathbb{P}^2$ when the support of Z is a complete intersection minus a point.

2000 Mathematics Subject Classification: 13D40, 13D02, 13H10, 14A15

1. Introduction

Let k be an algebraically closed field of characteristic zero and $R = k[x, y, z]$. If $I = (F, G) \subseteq R$ is a complete intersection of type (a, b), and if $I = \sqrt{I}$, then I is the defining ideal of ab points in $\mathbb{P}^2$. We shall say such a set X is a complete intersection of points of type (a, b), and we will denote it by $CI(a, b)$. Complete intersections of points satisfy the Cayley-Bacharach property (see [8]); more precisely:

Theorem 1.1. *Let $X = CI(a, b) \subseteq \mathbb{P}^2$ and let $P \in X$ be any point. Then the Hilbert function of $Y = X\backslash\{P\}$ is given by*

$$H_Y(t) = \min\{H_X(t), |X| - 1\} \quad \textit{for all } t \geq 0$$

where H_X is the Hilbert function of X.

Since the function H_X depends only on the type (a, b) of X, we see that H_Y only depends upon the type as well.

The theme of this paper is to understand Theorem 1.1 in the context of fat points. Let us be more specific about the problem. Let $P_1, \ldots, P_{ab}$ be the ab points of the complete intersection $X = CI(a, b)$, and let $m \in \mathbb{N}^+$. Then the ideal

$$I_Z = I_{P_1}^m \cap \cdots \cap I_{P_{ab}}^m \quad \text{where } P_i \leftrightarrow I_{P_i}$$

defines a homogeneous fat point scheme $Z \subseteq \mathbb{P}^2$ whose support is the complete intersection X. Using [18, Lemma 5, Appendix 6] we have that $I_Z = I_X^m$. Since I_Z is a power of a complete intersection, H_Z depends only upon the type (a, b) and m (see [11] for details). We will sometimes write $\{CI(a, b); m\}$ for Z.

Now let $P \in X$ be any point and consider the ideal

$$I_Y = I_{P_1}^m \cap \cdots \cap \widehat{I_P^m} \cap \cdots \cap I_{P_{ab}}^m.$$

The ideal I_Y defines the subscheme $Y \subseteq Z$ formed by removing the fat point (P, m) from Z. Note that the support of Y is a complete intersection minus a point. We shall denote Y by $\{CI(a,b); m\}\backslash\{(P,m)\}$. Because the Hilbert function of Z depends only upon (a,b) and m, it is natural to wonder if an analog of Theorem 1.1 holds for Y; that is, does the Hilbert function of Y depend only upon H_Z?

Examples in [2, 3, 9, 10, 11] show that the answer to this question is no. In keeping with the theme of these Proceedings, the Hilbert function of Y has the unexpected property that different constructions of the underlying complete intersection, e.g., whether or not the forms defining $CI(a,b)$ are irreducible, can result in different Hilbert functions. As well, examples can be constructed where H_Y depends upon what point P is removed from the support X.

To study the invariants associated to Y it is therefore necessary to introduce extra conditions on the underlying support. To this end, we introduce some relevant notation. Suppose $I = (F, G)$, with $\deg F = a$ and $\deg G = b$, defines a complete intersection of points of type (a,b). If both F and G are irreducible, then we say such a set of points is of type $CI_{gen}(a,b)$, and we sometimes denote this set by X_{gen}. If F and G are the product of linear forms, i.e., $F = L_1 \cdots L_a$ and $G = L'_1 \cdots L'_b$, then we say such a set of points is of type $CI_{grid}(a,b)$, and we denote sets of point of this type by X_{grid}. (The subscript *grid* is used to denote the fact that the points of X are the points of intersection in the grid formed by the lines defined by the L_is and L'_js.) We write $Z_{gen} = \{CI_{gen}(a,b); m\}$ to denote the homogeneous fat point scheme with $\mathrm{Supp}(Z_{gen}) = X_{gen}$ of multiplicity m, and $Y_{gen} = \{X_{gen}; m\}\backslash\{(P,m)\}$ to be the homogeneous fat point scheme with $\mathrm{Supp}(Y_{gen}) = X_{gen}\backslash\{P\}$. The schemes Z_{grid} and Y_{grid} are defined similarly. If X, Z, or Y is written without a subscript, then the results depend only on the type.

We describe the results in this paper. In §2, we investigate the invariant $\alpha(I_Y) := \min\{i \mid (I_Y)_i \neq 0\}$, the smallest degree of a form passing through Y. We begin by showing that computing $\dim_k (I_Y)_t$ is equivalent to computing $\dim_k[(L^m) \cap I_Z]_{t+m}$ for an appropriate linear form L. We then use this result to show that if $a < b$, then $\alpha(I_Y) = \alpha(I_Z)$ regardless of the construction of the support. A conjecture on the behavior of $\alpha(I_Y)$ when $a = b$ is given at the end of the section. In §3, we restrict to double points, that is, when $m = 2$. It is shown that when $a \leq b$, except in some special cases, Y_{gen} and Y_{grid} have the same graded Betti numbers. We also describe these numbers in terms of a and b.

We now survey known results. Some of the invariants of Y were already known in the case that the complete intersection lies on a curve of small degree, i.e., $1 \leq a \leq 3$. When $a = 1$, the support of the set of fat points lies on a line. Thus H_Y can be determined from a result of Davis and Geramita [7]. If the complete intersection lies on a conic in $\mathbb{P}^2$, i.e., $a = 2$, then H_Y follows from Catalisano [5] (in the case of smooth conics) and Harbourne [14] (for arbitrary conics). When $a = 3$, the support lies on a plane cubic (possibly reducible and nonreduced) in $\mathbb{P}^2$.

The function H_Y follows from [15]. As well, [14] contains results on the resolutions of fat points in $\mathbb{P}^2$ lying on plane curves of degree at most three. Fat points on other complete intersections in $\mathbb{P}^2$ were studied in [2, 3, 9, 10]. The paper [11] initiated an investigation of this problem for complete intersections of points in $\mathbb{P}^n$.

2. The form of smallest degree passing through Y

If $Z \subseteq \mathbb{P}^2$ is any fat point scheme, then the numerical character $\alpha(I_Z) := \min\{i \mid (I_Z)_i \neq 0\}$ is the smallest degree of a form passing through Z. As described in [13, page 89], if one can compute $\alpha(Z)$ for any Z, then one can compute H_W for any specific fat point scheme W (see also [12] in this volume). Therefore, it is of interest to study $\alpha(I_Y)$ if $Y = Z\backslash\{(P,m)\}$ and (P,m) is any fat point of $Z = \{CI(a,b);m\}$.

We begin by giving an alternative way to compute $\dim_k(I_Y)_t$.

Lemma 2.1. *Let $X = CI(a,b) \subseteq \mathbb{P}^2$. Fix an $m \in \mathbb{N}^+$ and $P \in X$, and set $Z = \{X;m\}$ and $Y = Z\backslash\{(P,m)\}$. Then there exists a linear form $L \in R$ that satisfies the following properties:*

1. *the line defined by L passes through P but not through any $Q \in X\backslash\{P\}$.*
2. *if $I_X = (F,G)$, then $L \nmid F$ or G.*
3. $\dim_k(I_Y)_t = \dim_k[(L^m) \cap I_Z]_{t+m}$ *for all $t \in \mathbb{N}$.*

Proof. There are an infinite number of lines that pass through P. However, only a finite number of these lines will also pass through a point $Q \in X\backslash\{P\}$. Furthermore, only a finite number of linear forms can divide F or G. So, it is clear that a linear form $L \in R$ can be found that satisfies (1) and (2).

Let L be a linear form that satisfies (1) and (2), and consider the following short exact sequence:

$$0 \longrightarrow R/(I_Z : (L^m))(-m) \xrightarrow{\times L^m} R/I_Z \longrightarrow R/(I_Z, L^m) \longrightarrow 0.$$

Because $(I_Z : (L^m)) = I_Y$, the short exact sequence implies that

$$\dim_k(I_Y)_t = \dim_k R_t + \dim_k(I_Z)_{t+m} - \dim_k(I_Z, L^m)_{t+m} \quad \text{for all } t \in \mathbb{N}.$$

Now $\dim_k(I_Z, L^m)_{t+m} = \dim_k(I_Z)_{t+m} + \dim_k(L^m)_{t+m} - \dim_k((L^m) \cap I_Z)_{t+m}$. Substituting this expression into the above identity, and also using the fact that $\dim_k(L^m)_{t+m} = \dim_k R_t$ then gives us the conclusion

$$\dim_k(I_Y)_t = \dim_k((L^m) \cap I_Z)_{t+m} \quad \text{for all } t \in \mathbb{N}.$$

□

It follows from Lemma 2.1 that to determine H_Y, it is enough to determine the Hilbert function of $R/((L^m) \cap I_Z)$. Lemma 2.1 will allow us to show that $\alpha(I_Y) = \alpha(I_Z)$ when $a < b$. Note that since $I_Z = I_X^m$, it follows that $\alpha(I_Z) = ma$, and because $I_Z \subseteq I_Y$, $\alpha(I_Y) \leq \alpha(I_Z)$.

Theorem 2.2. *Let $X = CI(a,b) \subseteq \mathbb{P}^2$. Fix an $m \in \mathbb{N}^+$ and $P \in X$, and set $Z = \{X; m\}$ and $Y = Z \backslash \{(P,m)\}$.*

1. *If $a < b$, then*
$$\alpha(I_Y) = \alpha(I_Z) = ma.$$
2. *If $m \geq 2$ and $m(a+1) - 3 < (m-1)b$, then*
$$H_Y(t) = H_Z(t) \text{ if and only if } t \leq ma + b - 3,$$
or equivalently,
$$\alpha(I_Y/I_Z) = ma + b - 2$$
when we consider I_Y/I_Z as an ideal of R/I_Z.

Proof. The proof of both statements will be by induction on m. We therefore set $Z_m := \{CI(a,b); m\}$ and $Y_m := \{CI(a,b); m\} \backslash \{(P,m)\}$ for each $m \in \mathbb{N}^+$.

(1) If $m = 1$, then the conclusion follows from Theorem 1.1. So, suppose $m \geq 2$. We need to show that $\dim_k (I_{Y_m})_{ma-1} = 0$. If L is the linear form of Lemma 2.1, then it shall be enough to show that $((L^m) \cap I_{Z_m})_{ma+m-1} = 0$ since Lemma 2.1 (3) then implies
$$\dim_k (I_{Y_m})_{ma-1} = \dim_k ((L^m) \cap I_{Z_m})_{ma+m-1} = 0.$$

So, suppose $K \in ((L^m) \cap I_{Z_m})_{ma+m-1}$, and hence $K = L^m A \in (I_{Z_m})_{ma+m-1}$. Because $a < b$, we have $ma + m - 1 = m(a+1) - 1 \leq mb - 1 < mb$. So
$$(I_{Z_m})_{ma+m-1} = (F^m, F^{m-1}G, \ldots, FG^{m-1}, G^m)_{ma+m-1} = F(I_{Z_{m-1}})_{(m-1)a+m-1}.$$
We thus have $L^m A = FH$ with $H \in I_{Z_{m-1}}$. Now L does not divide F, so $H = L^m H' \in (I_{Z_{m-1}})_{(m-1)a+m-1} \subseteq (I_{Y_{m-1}})_{(m-1)a+m-1}$. Because L is a nonzero divisor on $R/(I_{Y_{m-1}})$, this means that $H' \in (I_{Y_{m-1}})_{(m-1)a-1}$. By induction, we now have $H' = 0$, and hence $K = 0$ as desired.

(2) ($\Rightarrow$) Let $t \geq ma + b - 2$. It follows from Theorem 1.1 that there exists a form H such that $H \in (I_{Supp(Y)})_{a+b-2} \backslash (I_X)_{a+b-2}$. Let K be any linear form that defines a line that does not pass through any point of X. Then $F^{m-1}HK^{t-ma-b+2} \in (I_Y)_t \backslash (I_Z)_t$, thus giving $H_Y(t) \neq H_Z(t)$.

($\Leftarrow$) Suppose $m \geq 2$ and $m(a+1) - 3 < (m-1)b$, and let L be the linear form of Lemma 2.1. Our goal is to show that $((L^m) \cap I_{Z_m})_{t+m} = L^m (I_{Z_m})_t$ for each $t \leq ma + b - 3$. Then by Lemma 2.1 (3) we will have
$$\dim_k (I_{Y_m})_t = \dim_k ((L^m) \cap I_{Z_m})_{t+m} = \dim_k L^m (I_{Z_m})_t = \dim_k (I_{Z_m})_t$$
which implies $H_Z(t) = H_Y(t)$ for $t \leq ma + b - 3$.

Suppose $m = 2$. Since it is clear that $L^m(I_{Z_m})_t \subseteq ((L^m) \cap I_{Z_m})_{t+m}$, we need to show the reverse inclusion. So, suppose $K \in ((L^m) \cap I_{Z_m})_{t+m}$, and hence $K = L^m A \in (I_{Z_m})_{t+m}$. Because $t + m \leq ma + b - 3 + m = m(a+1) - 3 + b < (m-1)b + b = mb$ we have

$$(I_{Z_m})_{t+m} = (F^m, F^{m-1}G, \ldots, FG^{m-1}, G^m)_{t+m} = F(I_{Z_{m-1}})_{t+m-a}.$$

Hence $K = L^m A = FH$ with $H \in (I_{Z_{m-1}})_{t+m-a} \subseteq (I_{Y_{m-1}})_{t+m-a}$. Because $L \not| F$, we have $H = L^m H'$. Furthermore, since L is a nonzero divisor on $R/(I_{Y_{m-1}})$ we thus have $H' \in (I_{Y_{m-1}})_{t-a}$. Now $t-a \leq (m-1)a+b-3$. Since $m = 2$, $(I_{Y_{m-1}})_{t-a} = (I_{Z_{m-1}})_{t-a}$ by Theorem 1.1. So $FH' \in (I_{Z_m})_t$, and thus $K \in L^m(I_{Z_m})_t$.

If $m > 2$, then the proof is the same as the above argument, except that the induction hypothesis, instead of Theorem 1.1, is used to justify the fact that $(I_{Y_{m-1}})_{t-a} = (I_{Z_{m-1}})_{t-a}$. □

Remark 2.3. *Theorem* 2.2 (1) *was first conjectured to hold in* [3]. *Computer evidence suggests that statement* (2) *can be improved to:*

if $a < b$ *and* $m \in \mathbb{N}^+$, *then* $H_Z(t) = H_Y(t)$ *if and only if* $t \leq ma+b-3$.

Notice that the proof of the ($\Rightarrow$) *direction of* (2) *already proves one direction of this result. Furthermore, when* $m = 1$ *this statement is true by Theorem* 1.1*; if* $m = 2$ *then this follows from [3, Theorem 4.4]. The Hilbert function* H_Y *for* $m = 3$ *is studied in* [10].

When $a = b$, the value $\alpha(I_Y)$ may be different than $\alpha(I_Z)$. This was first shown in [3, Proposition 3.6]:

Proposition 2.4. *Let* $X = CI(a,a) \subseteq \mathbb{P}^2$. *Let* $m \in \mathbb{N}^+$, $P \in X$, *and set* $Z = \{X; m\}$ *and* $Y = Z \backslash \{(P,m)\}$. *If* $m > a^2 - a - 1$, *then* $\alpha(I_Y) < \alpha(I_Z)$.

Under the hypotheses of the previous proposition, if we set

$$m_Y := \max\{m \mid \alpha(I_Y) = \alpha(I_Z)\},$$

the above result then implies that $m_Y \leq a^2 - a - 1$. Surprisingly, the value of m_Y seems to depend on the construction of the underlying complete intersection. Using computer evidence generated by CoCoA [6], we have made the following conjecture on the value of $\alpha(I_Y)$.

Conjecture 2.5. *Let* $X = CI(a,a) \subseteq \mathbb{P}^2$, *and let* $Y = \{X; m\} \backslash \{(P,m)\}$ *for any* $P \in X$.

1. $\alpha(I_{Y_{grid}}) = ma$ *if and only if* $m \leq (a-1)^2$. *That is,* $m_{Y_{grid}} = (a-1)^2$.
2. $\alpha(I_{Y_{gen}}) = ma$ *if and only if* $m \leq a^2 - a - 1$. *That is,* $m_{Y_{gen}} = a^2 - a - 1$.

Remark 2.6. *B. Harbourne* [16] *has shown us that if* $a > 3$ *and* $X = X_{gen} = CI_{gen}(a,a) \subseteq \mathbb{P}^2$, *then* $\frac{a^2-a}{2} \leq m_{Y_{gen}} < a^2 - a$, *and if* $X = X_{grid} = CI_{grid}(a,a)$, *then* $a - 1 \leq m_{Y_{grid}} < a^2 - a - 2$. *This conjecture will be further explored in* [1].

3. Graded Betti numbers

Let $X = CI(a,b) \subseteq \mathbb{P}^2$. By [3, Theorem 4.4] $Y_{grid} = \{X_{grid}; 2\} \backslash \{(P,2)\}$ and $Y_{gen} = \{X_{gen}; 2\} \backslash \{(P,2)\}$ have the same Hilbert function. However, Example 4.6 in [3] showed that they may not have the same graded Betti numbers. Here we show that Y_{gen} and Y_{grid} always have the same graded Betti numbers if $a \leq b$ except in the case that $(a,b) = (2,b)$ with $b \geq 3$.

We shall require the following lemmas. The first is a consequence of Theorem 1.1.

Lemma 3.1. *Let $X = CI(a,b) \subseteq \mathbb{P}^2$ with defining ideal $I_X = (F,G)$. Let $P \in X$ be any point and set $Y = X \backslash \{P\}$. Then there exists a form H with* $\deg H = a+b-2$ *such that $I_Y = (F,G,H)$. Furthermore, the graded minimal free resolution of I_Y has the form*

$$0 \to R^2(-a-b+1) \to R(-a) \oplus R(-b) \oplus R(-a-b+2) \to I_Y \to 0.$$

We define $\Delta H(t) := H(t) - H(t-1)$ with $H(t) = 0$ if $t < 0$. More generally, $\Delta^d H(t) := \Delta^{d-1} H(t) - \Delta^{d-1} H(t-1)$.

Lemma 3.2. *Let $X \subseteq \mathbb{P}^2$ be any zero-dimensional scheme with defining ideal I_X and Hilbert function H_X. If α_t, respectively β_t, denotes the number generators, respectively the number of syzygies, of degree t of I_X, then*

$$-\Delta^3 H_X(t) = \alpha_t - \beta_t.$$

Furthermore, we have the bounds

$$\max\{0, -\Delta^3 H_X(t)\} \leq \alpha_t \leq -\Delta^2 H_X(t)$$

and

$$\max\{0, -\Delta^3 H_X(t)\} \leq \beta_t \leq -\Delta^2 H_X(t-1).$$

Proof. See [4] and [17]. □

We now come to the main theorem of this section.

Theorem 3.3. *In $\mathbb{P}^2$ consider the two schemes of double points*

$$Y_{gen} = \{CI_{gen}(a,b); 2\} \backslash \{(P,2)\} \text{ and } Y_{grid} = \{CI_{grid}(a,b); 2\} \backslash \{(P,2)\}$$

with $a \leq b$. We write Y to indicate both Y_{grid} and Y_{gen}.

1. *If $a = 1$ and $b > 1$, then the resolution of Y has the form*

$$0 \to R(-b-1) \oplus R(-2b+1) \to R(-2) \oplus R(-b-2) \oplus R(-2b+2) \to I_Y \to 0.$$

2. *If $a = 2$ and $b \geq 3$, then*

(*i*) *the resolution of* $I_{Y_{grid}}$ *has the form*

$$0 \to R(-2b) \oplus R(-2b-1) \oplus R^2(-b-3) \to$$
$$\to R(-4) \oplus R(-2b) \oplus R^2(-b-2) \oplus R(-2b+1) \to I_Y \to 0.$$

(*ii*) *the resolution of* $I_{Y_{gen}}$ *has the form*

$$0 \to R(-2b-1) \oplus R^2(-b-3) \to$$
$$\to R(-4) \oplus R^2(-b-2) \oplus R(-2b+1) \to I_{Y_{gen}} \to 0.$$

3. If $2 < a < b$*, then the resolution of* I_Y *has the form*

$$0 \to R(-a-2b+1) \oplus R(-a-2b+2) \oplus R^2(-2a-b+1) \to$$
$$\to R(-2a) \oplus R(-2b) \oplus R(-a-2b+3) \oplus R(-a-b) \oplus R(-2a-b+2) \to I_Y \to 0.$$

4. If $1 < a = b$*, then the resolution of* I_Y *has the form*

$$0 \to R^3(-3a+1) \to R^3(-2a) \oplus R(-3a+3) \to R/I_Y \to 0.$$

Proof. (1) If $a = 1$ and $1 < b$, then $Y_{gen} = Y_{grid} = \{CI(1, b-1); 2\}$, in which case the result follows from [7].

(2) If $a = 2$ and $b \geq 3$, then, from [2, Proposition 4.7], $Y_{grid} = \{CI_{grid}(2, b); 2\} \backslash \{(P, 2)\}$ has a minimal free resolution of type (*i*). Because

$$Y_{gen} = \{CI_{gen}(2, b) \backslash \{(P, 2)\}$$

lies on an irreducible conic, we can use [5] to verify that the minimal free resolution has the form given in (*ii*). Therefore Y_{grid} and Y_{gen} have different graded Betti numbers when $(a, b) = (2, b)$ with $b \geq 3$.

(3) Suppose $2 < a < b$. Using [2, Proposition 4.7]

$$Y_{grid} = \{CI_{grid}(a, b); 2\} \backslash \{(P, 2)\}$$

has the same graded Betti numbers as a partial intersection $Y^{p.i.}$ of type (p, q) where $p = (2b, 2b-2, b, b-1)$ and $q = (a-1, 1, a-1, 1)$. Hence, the minimal graded free resolution of $I_{Y_{grid}}$ has the form given in (3).

Since Y_{grid} and Y_{gen} are both zero-dimensional schemes of $\mathbb{P}^2$, and from [3] they have the same Hilbert function, to show that they have the same minimal graded Betti numbers, it suffices to show that the number of generators of $I_{Y_{gen}}$ and their degrees are the same as those of $I_{Y_{grid}}$. Then the degrees of the syzygies are a consequence of Lemma 3.2.

Let F and G denote the two irreducible forms of degree a and b, respectively, that generate $CI_{gen}(a, b)$. We shall consider the cases $b \geq 2a-2$ and $b < 2a-2$ separately.

If $b \geq 2a-2$, from [3, Theorem 4.4] we get

$$\Delta^3 H_{Y_{gen}}(t) = \begin{cases} 1 & t=0 \\ 0 & 1 \leq t \leq 2a-1 \\ -1 & t=2a \\ 0 & 2a+1 \leq t \leq a+b-1 \\ -1 & t=a+b \\ 0 & a+b+1 \leq t \leq 2a+b-3 \\ -1 & t=2a+b-2 \\ 2 & t=2a+b-1 \\ 0 & 2a+b \leq t \leq 2b-1 \\ \begin{cases} -2 \text{ if } a=3 \\ -1 \text{ if } a \neq 3 \end{cases} & t=2b \\ \begin{cases} 1 \text{ if } a=3 \\ 0 \text{ if } a \neq 3 \end{cases} & t=2b+1, 2b+2 \\ 0 & 2b+3 \leq t \leq a+2b-4 \\ -1 & t=a+2b-3 \\ 1 & t=a+2b-2, a+2b-1 \\ 0 & t \geq a+2b. \end{cases}$$

Let α_i, respectively β_i, denote the number of generators, respectively syzygies, in degree i. So, if $a=3$, we want to show that $\alpha_{2a} = \alpha_{a+b} = \alpha_{2a+b-2} = 1$, $\alpha_{2b} = 2$, and $\alpha_i = 0$ otherwise, and if $a \neq 3$, then we want to show $\alpha_{2a} = \alpha_{a+b} = \alpha_{2a+b-2} = \alpha_{2b} = \alpha_{a+2b-3} = 1$ and $\alpha_i = 0$ otherwise.

It is clear that $\alpha_{2a} = \alpha_{a+b} = 1$ for all $a \geq 3$, and $\alpha_i = 0$ for all other $i \leq a+b$. Since $F^2, FG \in I_{Y_{gen}}$ with $\deg F^2 = 2a$ and $\deg FG = a+b$, we can assume that these forms are the minimal generators of degrees $2a$ and $a+b$, respectively.

By Lemma 3.2 we have $0 \leq \beta_i \leq 1$ for $a+b < i \leq 2a+b-2$. If $\beta_i = 1$ in this interval, there must be a syzygy among the generators of degree $< 2a+b-2$, i.e., there exist f_1 and f_2 such that $f_1F^2 + f_2FG = 0$ with f_1 and f_2 forms of suitable degrees. But this implies that $f_1F + f_2G = 0$, i.e., that this syzygy must have degree $2a+b$ because F and G form a regular sequence. This contradicts the fact that $i \leq 2a+b-2$. So, using Lemma 3.2, $\beta_i = \alpha_i = 0$ for $a+b < i < 2a+b-2, \alpha_{2a+b-2} = 1$, and $\beta_{2a+b-2} = 0$.

Let H be the form of degree $a+b-2$ of Lemma 3.1. Then $FH \in I_{Y_{gen}}$ and $\deg FH = 2a+b-2$. Moreover, since FH is not in the ideal generated by F^2 and FG, we can take FH to be the minimal generator of degree $2a+b-2$.

To compute α_i for $2a+b-1 \leq i \leq 2b$, we again compute β_i on this interval. Note that there is a syzygy of degree i among F^2, FG, FH, that is, $f_1F^2 + f_2FG + f_3FH = 0$, if and only if there is a syzygy among F, G, and H of degree $i-a$, that is, $f_1F + f_2G + f_3H = 0$. But F, G, H generate $I_{\mathrm{Supp}(Y_{gen})}$, and by Lemma 3.1, there are only two minimal syzygies of degree $a+b-1$. But this means that $\beta_{2a+b-1} = 2$ and $\beta_i = 0$ for $i = 2a+b, \ldots, 2b$. Then Lemma 3.2 gives $\alpha_i = 0$ for all $i = 2a+b-1, \ldots, 2b-1$, and $\alpha_{2b} = 2$ if $a=3$, and $\alpha_{2b} = 1$ if $a \neq 3$.

If $a=3$, then we are done since the regularity of $I_{Y_{gen}}$ equals $a+2b-3$ (this can be read off of the Hilbert function), so there are no generators of higher degree.

So, suppose $a \neq 3$ and compute α_i for $2b+1 \leq i \leq a+2b-3$. Since $G^2 \in I_{Y_{gen}}$ and $\deg G^2 = 2b$, we can take G^2 to be the minimal generator of degree $2b$. Using Lemma 3.2 we have $0 \leq \beta_i \leq 1$ for $2b+1 \leq i \leq a+2b-3$. If $\beta_i = 1$ in this interval, then $f_1F^2 + f_2FG + f_3FH + f_4G^2 = 0$ with f_1, f_2, f_3, f_4 forms of suitable degrees. But this means $G(f_4G) = FK$ for some suitable form K. This contradicts the fact that F and G form a regular sequence. Hence, $\beta_i = \alpha_i = 0$ if $i = 2b+1, \ldots, a+2b-4$, $\beta_{a+2b-3} = 0$ and $\alpha_{a+2b-3} = 1$.

Again, since $\operatorname{reg}(I_{Y_{gen}}) = a+2b-3$, there can be no other generators of higher degree, thus completing the case that $b \geq 2a-2$.

If $b < 2a-2$ then

$$\Delta H_{Y_{gen}}(t) = \begin{cases} t+1 & 0 \leq t \leq 2a-1 \\ 2a & 2a \leq t \leq a+b-1 \\ 3a+b-1-t & a+b \leq t \leq 2b-1 \\ 3a+3b-2-2t & 2b \leq t \leq 2a+b-3 \\ a+2b-1-t & 2a+b-2 \leq t \leq a+2b-4 \\ 1 & t = a+2b-3 \\ 0 & t \geq a+2b-2\,. \end{cases}$$

Using an argument similarly to the one given above, we get the conclusion.

(4) It was shown in [3] that if $a = b$, then Y_{grid} and Y_{gen} have the same Hilbert function, namely:

$$\Delta H_{Y_{grid}}(t) = \Delta H_{Y_{gen}}(t) = \begin{cases} t+1 & 0 \leq t \leq 2a-1 \\ 2(3a-t-1) & 2a \leq t \leq 3a-4 \\ 3 & t = 3a-3 \\ 0 & t \geq 3a-2. \end{cases}$$

Now arguing as we did above, we get that both Y_{grid} and Y_{gen} have a minimal resolution of type:

$$0 \to R^3(-3a+1) \to R^3(-2a) \oplus R(-3a+3) \to R/I_Y \to 0.$$

□

Acknowledgments. The authors would like to thank the organizers of "Projective Varieties with Unexpected Properties" for an interesting conference, and for the opportunity to visit beautiful Siena. The second author also would like to acknowledge the financial support of Natural Sciences and Engineering Research Council of Canada.

References

[1] M. Buckles, E. Guardo, B. Harbourne, A. Van Tuyl, Some results on forms of least degree in ideals of fat points in special position, in preparation.

[2] M. Buckles, E. Guardo, A. Van Tuyl, Fat points on a grid in $\mathbb{P}^2$, Le Matematiche (Catania) **55** (2000), 169–189 (2001).

[3] M. Buckles, E. Guardo, A. Van Tuyl, Fat points on a generic almost complete intersection, Le Matematiche (Catania) **55** (2000), 191–202 (2001).

[4] G. Campanella, Standard bases of perfect homogeneous polynomial ideals of height 2, J. Algebra **101** (1986), 47–60.

[5] M.V. Catalisano, "Fat" points on a conic, Comm. Algebra **19** (1991), 2153–2168.

[6] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra, Available at `http://cocoa.dima.unige.it`

[7] E.D. Davis, A.V. Geramita, The Hilbert function of a special class of 1-dimensional Cohen-Macaulay graded algebras, The curves seminar at Queen's, Vol. III (Kingston, Ont., 1983), Exp. No. H, 29 pp., Queen's Papers in Pure and Appl. Math., **67** Queen's Univ., Kingston, ON, 1984.

[8] A.V. Geramita, M. Kreuzer, L. Robbiano, Cayley-Bacharach schemes and their canonical modules, Trans. Amer. Math. Soc. **339** (1993), 163–189.

[9] E. Guardo, Schemi di "Fat Points", Ph.D. Thesis, Università di Messina (2000).

[10] E. Guardo, Subschemes of fat points on complete intersections and partial intersections, (2004) Preprint.

[11] E. Guardo, A. Van Tuyl, Powers of complete intersections: graded Betti numbers and applications (2004), preprint. `math.AC/0409090`

[12] B. Harbourne, The (unexpected) importance of knowing alpha, (2004), in this volume.

[13] B. Harbourne, Problems and progress: a survey on fat points in $\mathbb{P}^2$, in *Zero-dimensional schemes and applications (Naples, 2000)*, 85–132, Queen's Papers in Pure and Appl. Math., **123** Queen's Univ., Kingston, ON, 2002.

[14] B. Harbourne, Free resolutions of fat point ideals on $\mathbb{P}^2$, J. Pure Appl. Algebra **125** (1998) 213–234.

[15] B. Harbourne, Anticanonical rational surfaces, Trans. Amer. Math. Soc. **349** (1997) 1191–1208.

[16] B. Harbourne, personal communication.

[17] R. Maggioni, A. Ragusa, Construction of smooth curves of $\mathbb{P}^3$ with assigned Hilbert function and generators' degrees, Le Matematiche (Catania) **42** (1987) 195–209 (1989).

[18] O. Zariski, P. Samuel, *Commutative Algebra, Volume II.* D. Van Nostrand Company, Princeton, NJ, 1960.

Elena Guardo
Dipartimento di Matematica e Informatica
Viale A. Doria, 6 - 95100 - Catania, Italy

Email: guardo@dmi.unict.it

Adam Van Tuyl
Department of Mathematical Sciences
Lakehead University
Thunder Bay, ON P7B 5E1, Canada

Email: avantuyl@sleet.lakeheadu.ca