Workshop on computational and combinatorial commutative algebra
Fields Institute, Toronto, July 24, 2006

Introduction

The afternoon tutorials give you a chance to play around and experiment with *Macaulay 2*. Each tutorial begins with some needed definitions and results and ends with a list of references. Some of the initial problems ask you to prove some simple results, to give you a feeling for the material, while other problems ask you to program some simple procedures using *Macaulay 2*, to help you develop your *Macaulay 2* skills. The last batch of questions for each tutorial is a series of open questions, which are denoted by an asterisk. (If you come up with any ideas, we would love to hear them!)

Tutorial 1: Resolutions of edge ideals

Let $G = (V, E)$ be a simple graph, that is, a graph with no loops or multiple edges. If $V = \{x_1, \ldots, x_n\}$ are the vertices of G, by identifying the vertices with the variables of $R = k[x_1, \ldots, x_n]$, we can associate to G the monomial ideal $\mathcal{I}(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E\})$. The ideal $\mathcal{I}(G)$ is called the *edge ideal* of G. For example if $G = C_4$ is the 4-cycle, then the edge ideal is $\mathcal{I}(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1)$. The notion of an edge ideal was first introduced by Villarreal. Note that G can be viewed as a 1-dimensional simplicial complex where the edges are the facets. An edge ideal is a different way to associate to a simplicial complex a monomial ideal.

Among other things, in this tutorial we are interested in finding relationships between the invariants in the minimal graded resolution of $R/\mathcal{I}(G)$:

$$0 \to \bigoplus_j R(-j)^{\beta_{l,j}(R/\mathcal{I}(G))} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(R/\mathcal{I}(G))} \to R \to R/\mathcal{I}(G) \to 0$$

and the combinatorial properties of G.

The following definitions from graph theory will be useful. A *clique* of size n, denoted K_n, is a graph on n vertices such that there is an edge between every pair of vertices. If $G = (V, E)$ is a simple graph, and if $W \subseteq V$ is a subset of V, then the *induced graph on* W, denoted G_W, is the graph whose edge set is given by $\{e \in E \mid e \subseteq W\}$. The *complement* of G, denoted G^c, is the graph with the same vertex set as G, but whose edge set is given by the rule $\{x_i, x_j\} \in E_{G^c}$ if and only if $\{x_i, x_j\} \notin E_G$. A graph G is *chordal* if every cycle of length $n > 3$ has a chord. Here, if $\{x_1, x_2\}, \ldots, \{x_n, x_1\}$ are the n edges of a cycle of length n, we say the cycle has a chord in G if there exists two vertices x_i, x_j in the cycle such that $\{x_i, x_j\}$ is also an edge of G, but $\{x_i, x_j\}$ is not an edge of the cycle. To learn more about edge ideals and their resolutions, a good place to start would be [V, HVT2].

Exercise 1.1. Let $G = (V, E)$ be a simple graph. The *clique complex* is the set

$$\Delta(G) = \{W \subseteq V \mid \text{the induced graph } G_W \text{ is a clique}\}.$$

Show that $\Delta(G)$ is a simplicial complex.

Exercise 1.2. Since the edge ideal $\mathcal{I}(G)$ of a simple graph G is a squarefree monomial ideal, the ideal $\mathcal{I}(G)$ is also the Stanley-Reisner ideal of some simplicial complex. Let Δ be the simplicial complex defined by $\mathcal{I}(G)$ via the Stanley-Reisner correspondence. Show $\Delta = \Delta(G^c)$, that is, the clique complex of G^c.

Exercise 1.3. Write a *Macaulay 2* program that does the following: given a graph G, return the facets (the maximal faces) of $\Delta(G^c)$.

Hint. You may want to use the `SimplicialComplexes` package.

Exercise 1.4. An *independent set* of vertices of a graph $G = (V, E)$ is a subset $W \subseteq V$ such that G_W is the graph of isolated vertices. Show that if d is the cardinality of the largest independent set of vertices of G, then $\dim(R/\mathcal{I}(G)) = d$.

Hint. What do the maximal (with respect to inclusion) independent sets of vertices correspond to in $\Delta(G^c)$?

Exercise 1.5. Explain why $\beta_{1,2}(R/\mathcal{I}(G))$ is the number of edges, and $\beta_{1,j}(R/\mathcal{I}(G)) = 0$ if $j \neq 2$.

Exercise 1.6. Show that if $\beta_{i,j}(R/\mathcal{I}(G)) \neq 0$, then $i + 1 \leq j \leq 2i$.

Remark. When looking for formulas for $\beta_{i,j}(R/\mathcal{I}(G))$, this result says we only need to consider a specific range of j for each i.

Exercise 1.7. When $i = 2$, Eliahou-Villarreal [EV] showed that $\beta_{i,j}(R/\mathcal{I}(G))$ is given by the formula

$$\beta_{2,j}(R/\mathcal{I}(G)) = \begin{cases} \sum_{v \in V} \binom{\deg v}{2} - k_3(G) & \text{if } j = 3 \\ c_4(G^c) & \text{if } j = 4 \\ 0 & \text{otherwise.} \end{cases}$$

where $k_3(G)$ denotes the number of triangles of G (a triangle is also a 3-cycle), and $c_4(G^c)$ denotes the number of 4-cycles in G^c. Given a graph G, write a *Macaulay 2* program that counts the number of triangles of G. Write another program that counts the number of 4-cycles in G.

Hint. The program `BettiIJ` introduced in the lectures should be helpful.

Exercise 1.8. Suppose that I is an ideal generated in a single degree, say d. Recall that we say that I has a *linear resolution* if $\beta_{i,j}(R/I) = 0$ for all $j \neq i + d - 1$ and $i \geq 1$. Fröberg [F] proved the following result: $\mathcal{I}(G)$ has a linear resolution if and only if G^c is a chordal graph. Using this fact, write a *Macaulay 2* script that determines if a given graph G is chordal. Note that $(G^c)^c = G$.

***Exercise 1.9.** When $i \geq 3$, we have formulas for $\beta_{i,j}(R/\mathcal{I}(G))$ in only some special cases. Using *Macaulay 2*, try to come up with a conjecture of $\beta_{3,4}(R/\mathcal{I}(G))$. Compare your conjecture with the formula in [RVT]. Try finding other formulas for $\beta_{i,j}(R/\mathcal{I}(G))$.

*__Exercise 1.10.__ A *matching* of a graph is a set of pairwise disjoint edges of G. Let $\alpha'(G)$ denote the size of the largest matching in G. Then it was recently shown in [HVT1] that $\operatorname{reg}(R/\mathcal{I}(G)) \leq \alpha'(G)$. When $G = C_5$, the 5-cycle, then the upper bound is achieved, since the largest matching in C_5 consists of two edges, which equals the regularity. It would be interesting to find a family of graphs for which the upper bound is always achieved. Try to find a family of graphs using *Macaulay 2* for which the upper bound is achieved.

Remark. If G is a chordal graph, then there is an exact formula (see [HVT1]) for $\operatorname{reg}(R/\mathcal{I}(G))$ which is less than $\alpha'(G)$ in general, so you should limit you search to non-chordal graphs.

*__Exercise 1.11.__ Recall that a graph is *bipartite* if the vertex set V can be partitioned into two disjoint subsets $V = V_1 \cup V_2$, such that every edge of G has one vertex in V_1 and the other in V_2. Is there a formula for $\operatorname{reg}(R/\mathcal{I}(G))$ when G is bipartite? Use *Macaulay 2* to come up with a conjecture.

Remark. If G is a tree, then G is also a chordal graph, so there is formula for the regularity in this case. However, I'm not aware of any formula for the regularity when G is bipartite, but not a tree.

*__Exercise 1.12.__ Consider the projective dimension of $R/\mathcal{I}(G)$, denoted $\operatorname{pdim}(R/\mathcal{I}(G))$. Jacques and Katzman [JK] have shown that when G is a tree, then $\operatorname{pdim}(R/\mathcal{I}(G))$ can be computed recursively. However, I don't know of any relationship between this invariant and the combinatorial data of G. It would be nice to find some connection between the two, even for a special class of graphs (e.g. chordal).

Related References

[EV] Shalom Eliahou and Rafael H. Villarreal, *The second Betti number of an edge ideal*, (Hermosillo, 1998), Aportaciones Mat. Comun., vol. 25, Soc. Mat. Mexicana, México, 1999, pp. 115–119.

[F] Ralf Fröberg, *On Stanley-Reisner rings*, Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 57–70.

[HVT1] Huy Tài Hà and Adam Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their minimal graded free resolutions*, available at `arXiv:math.AC/0606539`.

[HVT2] ______, *Resolutions of square-free monomial ideals via facet ideals: a survey*, available at `arXiv:math.AC/0604301`.

[JK] Sean Jacques and Mordechai Katzman, *The Betti numbers of forests*, available at `arXiv:math.AC/0401226`.

[RVT] Mike Roth and Adam Van Tuyl, *On the linear strand of an edge ideal*, available at `arXiv:math.AC/0411181`.

[V] Rafael H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 238, Marcel Dekker Inc., New York, 2001.

2. **Tutorial 2: Componentwise linear ideals**

In this tutorial, we will explore componentwise linear ideals. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field k. For a homogeneous ideal I, we write (I_d) to denote the ideal generated by all degree d elements of I. Note that (I_d) is different from I_d, which is usually used to denote the vector space of all degree d elements of I.

Recall that if I is an ideal generated in a single degree, say e, then we say that I has a *linear resolution* if $\beta_{i,j}(R/I) = 0$ for all $j \neq i + e - 1$ and $i \geq 1$. Herzog and Hibi introduced the following definition in [HH]; a homogeneous ideal I is *componentwise linear* if (I_d) has a linear resolution for all d.

Set $[n] := \{1, \ldots, n\}$. For a nonempty subset $J = \{j_1, \ldots, j_t\} \subseteq [n]$, we define $\mathfrak{m}_J := (x_{j_1}, \ldots, x_{j_t})$. In this tutorial, we will focus primarily on understanding when monomial ideals of the form

$$(\ddagger) \qquad I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s} \text{ with } J_i \subseteq [n] \text{ and } a_i \in \mathbb{Z}^+,$$

are componentwise linear. These ideals arise naturally, for example, in the study of fat points, tetrahedral curves, and Alexander duality of squarefree monomial ideals (see references in [FVT1]).

Exercise 2.1. Let $d = \operatorname{reg}(I)$. Prove that if $e \geq d$, then (I_e) has linear resolution.

Hint. See [EG].

Exercise 2.2. Prove that if I has a linear resolution, then I is componentwise linear. Show, through an example, that the converse is not necessarily true, i.e., there exist componentwise linear ideals that do not have a linear resolution.

Exercise 2.3. Let $J \subseteq [n]$. Prove that for any $a \geq 1$, the ideal $I = \mathfrak{m}_J^a$ has a linear resolution (and thus, is componentwise linear). Find a formula for all the graded Betti numbers of I.

Exercise 2.4. Consider the ideal $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_1)$. Prove that I is *not* componentwise linear.

The following exercises will describe how to use *Macaulay 2* to determine if a monomial ideal of the form $(\ddagger)$ is componentwise linear.

Exercise 2.5. Write a *Macaulay 2* script that given s subsets $J_1, \ldots, J_s \subseteq [n]$ and s positive integers, returns the ideal of the form $(\ddagger)$.

Exercise 2.6. Write a *Macaulay 2* script that given a monomial ideal I and an integer d, returns the ideal (I_d).

Hint. How do the generators of (I_d) relate to the generators of $(x_1, \ldots, x_n)^d \cap I$? Alternatively, the command `truncate` might be useful.

Exercise 2.7. Using the previous two exercise, write a *Macaulay 2* script to test whether an ideal of the form $(\ddagger)$ is componentwise linear.

Hint. You may want to make use of Exercise 2.1, and the script for checking if an ideal has a linear resolution given in the lecture.

Exercise 2.8. If I is generated by squarefree monomials, let $I_{[d]}$ denote the ideal generated by the squarefree monomials of degree d of I. Proposition 1.5 in [HH] shows that when I is squarefree, then I is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all d.

Adapt your program in Exercise 2.7 to write a new program for the squarefree monomial case (i.e., all the a_is equal 1) that makes use of Herzog and Hibi's result. Using the `timing` command, compare your two algorithms.

Exercise 2.9. There is an alternative way to determine if (‡) is componentwise linear without needing to computing resolutions (provided that the characteristic of k is zero). Conca [C] proved that I is componentwise linear if and only if I and its generic initial ideal $\mathrm{gin}(I)$ with respect to the graded reverse lexicographical order have the same number of generators. Write a new program to test that I is componentwise linear using this result. You will want to use the method to compute $\mathrm{gin}(I)$ as described in the lecture.

***Exercise 2.10.** Let I be an ideal of the form (‡). For what J_is and a_is will I be componentwise linear? (See [FVT1] for some known cases.)

***Exercise 2.11.** Let I be an ideal of the form (‡). Are there formulas for the regularity and the projective dimension of I that depend only upon the J_is and a_is? Note that this question is interesting whether or not I is componentwise linear.

***Exercise 2.12.** Let $G = (V, E)$ be any finite simple graph on n vertices. We associate to G an ideal of the form (‡) by setting $I_G := \bigcap_{\{x_i,x_j\}\in E}(x_i, x_j)$. For example, if G is the 4-cycle, i.e. G has edge set $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\}$, then $I_G = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_1)$. (For those of you familiar with the Alexander dual, you can show that I_G equals $\mathcal{I}(G)^\vee$, the Alexander dual of the edge ideal of G)

In [FVT2], it was shown that if G is chordal graph, then I_G is componentwise linear. Are there other families of graphs for which I_G is componentwise linear?

Given a graph G, and a vertex x_i of G, we can add a "whisker" to a G by adding an new vertex y_i and a new edge $\{x_i, y_i\}$. In [FH], it was shown that by adding "whiskers" to certain vertices of G, the new graph G' would have the property that $I_{G'}$ is componentwise linear. What other operations can we apply to G to make a graph G' with the property that $I_{G'}$ is componentwise linear?

***Exercise 2.13.** Consider three subsets $J_i \subseteq [n]$ and any three positive integers a_1, a_2, a_3. Find a formula for the graded Betti numbers of $I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \mathfrak{m}_{J_3}^{a_3}$ that depends only upon J_1, J_2, J_3 and a_1, a_2 and a_3.

Remark. This result would have an interesting application. Let P_1, P_2, P_3 be three points in generic position in $\mathbb{P}^n$ with $n \geq 2$. After a change of coordinates, we can assume that $P_1 = [1 : 0 : 0 : \cdots : 0], P_2 = [0 : 1 : 0 : \cdots : 0]$ and $P_3 = [0 : 0 : 1 : \cdots : 0]$. If we take $J_i = [n + 1]\backslash\{i\}$ for $i = 1, 2, 3$, then $I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \mathfrak{m}_{J_3}^{a_3}$ is the defining ideal of three fat points of multiplicity a_1, a_2, and a_3 whose support is the three generic points. An answer to this question would give us the minimal graded resolution for this fat point scheme. See [V] for case of two fat points.

Related References

[C] Aldo Conca, *Koszul homology and extremal properties of Gin and Lex*, Trans. Amer. Math. Soc. **356** (2004), no. 7, 2945–2961 (electronic).

[EG] David Eisenbud and Shiro Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), no. 1, 89–133.

[FH] Christopher A. Francisco and Huy Tài Hà, *Whiskers and sequentially Cohen-Macaulay graphs*, available at `arXiv:math.AC/0605487`.

[FVT1] Christopher A. Francisco and Adam Van Tuyl, *Some families of componentwise linear monomial ideals*, available at `arXiv:math.AC/0508589`.

[FVT2] ———, *Sequentially Cohen-Macaulay Edge Ideals*, available at `arXiv:math.AC/0511022`.

[HH] Jürgen Herzog and Takayuki Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.

[V] Giuseppe Valla, *Betti numbers of some monomial ideals*, Proc. Amer. Math. Soc. **133** (2005), no. 1, 57–63 (electronic).