The path ideal of a tree and its properties

by

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Abstract

Given a tree Γ, we consider the path ideal $I_t(\Gamma)$, that is, the ideal where every generator corresponds to a path of length t in Γ. When this path ideal is regarded as a facet ideal of a simplicial complex, that is, we view every generator of the path ideal as a facet of this simplicial complex, we show this simplicial complex is actually a simplicial tree. By using a property of a simplicial tree due to Faridi, we prove that $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay.

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Introduction

A ***tree*** is a simple graph in which any two vertices are connected by exactly one path. A ***directed tree*** is a tree whose edges have been assigned a direction. A ***path*** of a tree is a sequence of vertices $\{x_{i_1}, \ldots, x_{i_l}\}$ such that from each of its vertices there is an edge to the next vertex in the sequence. A path is also often denoted by $x_{i_1} \cdots x_{i_l}$. Throughout this project, a tree refers to a directed tree with edges whose orientation is away from the root, a designated vertex of the tree. Precise definitions will be given in Chapter 1. Using the definition of a path in a tree, we consider an ideal whose generators correspond to the paths of the same length in a tree. This ideal is the ***path ideal*** first introduced by Conca and De Negri [**3**]. Please see the precise definition below.

The goal of this project is to study the properties of the path ideal of a tree. The main result of this project is to show that the path ideal of a tree is sequentially Cohen-Macaulay. Understanding further properties of the path ideal of any graph is the goal of my future research. Given below is an overview of the main results and structure of this project.

Let $V = \{x_1, x_2, \ldots, x_n\}$ be a finite set. Then a ***simplicial complex*** Δ is a subset of the power set of V, such that:

- $\{x_i\} \in \Delta$ for each $i = 1, \ldots, n$, and
- if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$.

An element of Δ is a ***face***. The maximal faces under inclusion are called ***facets***. We denote the simplicial complex Δ with facets $F_1, \ldots, F_s$ by

$$\Delta = \langle F_1, \ldots, F_s \rangle.$$

The ***subcomplex*** of a simplicial complex Δ is a simplicial complex generated by a subset of the facet set of Δ. That is, $\Delta' = \langle F_{i_1}, \ldots, F_{i_r} \rangle$ is a subcomplex of Δ, if $\{F_{i_1}, \ldots, F_{i_r}\} \subset \{F_1, \ldots, F_s\}$. A standard reference for simplicial complexes is Stanley's book [**22**].

A relationship between trees and simplicial complexes can be built by using ***square-free monomial ideals***, that is, an ideal generated by monomial, terms of the form $m = x_1^{a_1} \cdots x_n^{a_n}$ where $0 \le a_i \le 1$. The ***path ideal*** of the tree Γ is an ideal generated by all paths of length $t-1$ (length of a path = the number of vertices in this path -1), denoted $I_t(\Gamma)$:

$$I_t(\Gamma) = (\{x_{i_1} \cdots x_{i_t} \mid x_{i_1} \cdots x_{i_t} \text{ is a path of length } t-1 \text{ in } \Gamma\ \}).$$

When $t = 2$, we call this ideal the ***edge ideal***, denoted $I(\Gamma)$. Recall that the definition of an edge ideal for any graph is an ideal generated by all edges in a graph G and denoted $I(G) = (\{x_i x_j \mid \{x_i, x_j\} \text{ is an edge of the graph } G\})$ (see [**24**]). This ideal is a square-free monomial ideal. The path ideal was first defined by Conca and De Negri in [**3**] and P. Brumatti and A.F. da Silva use this notion to study the cycle graph (see [**1**]).

We define the ***facet ideal*** of a simplicial complex Δ to be $\mathcal{I}(\Delta)$, the ideal generated by all square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_t}$, where $\{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ is a facet of Δ. Thus

$$\mathcal{I}(\Delta) = (\{x_{i_1} \ldots x_{i_t} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\} \text{ is a facet of } \Delta\}).$$

This notion was first defined by Sara Faridi in [**10**] and studied in [**11**],[**12**].

An example is illustrated in Figure 1 and Figure 2:

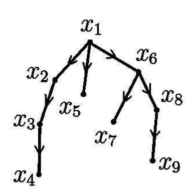

FIGURE 1

Given a tree Γ as in Figure 1, the root is x_1. We can find the path ideal generated by all paths of length 3 (i.e. there are 4 vertices in every path)

$$I_4(\Gamma) = (x_1x_2x_3x_4, x_1x_6x_8x_9).$$

The generators of $I_t(\Gamma)$ can also be viewed as the facets of a simplicial complex. For example, with Γ and $I_4(\Gamma)$ as above, consider the simplicial complex:

$$\Delta = \langle\{x_1, x_2, x_3, x_4\}, \{x_1, x_6, x_8, x_9\}\rangle.$$

We can visualize this simplicial complex as Figure 2:

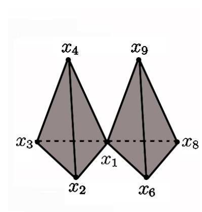

FIGURE 2

This simplicial complex Δ consists of two filled tetrahedrons. All the vertices, edges, triangles and the tetrahedrons themselves are the faces of this simplicial complex. The two tetrahedrons are facets because they are the maximal faces.

This example shows how to build a relationship between a tree Γ and a simplicial complex Δ by using the path ideal $I_t(\Gamma)$. When we view the generators of the path ideal as all the facets of a simplicial complex, the path ideal of the tree Γ is just the facet ideal of the corresponding simplicial complex, i.e.

$$I_t(\Gamma) = \mathcal{I}(\Delta).$$

In this paper, we will prove $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay. We recall the relevant definitions here. An element $F \in R$ is a ***regular*** element on R/I if $\bar{F} = (F + I)$ is not a zero divisor of R/I. A sequence $F_1, \ldots, F_m$ of R is called a ***regular sequence*** on R/I if

(1) $\overline{F_1}$ is regular on R/I, and
(2) $\overline{F_i}$ is regular on $R/(I, F_1, \cdots, F_{i-1})$, for $i = 2, \ldots, m$.

All maximal regular sequence have same length, and any regular sequence can be extended to a maximal regular sequence [**6**, Corollary 18.10]. The **depth** of R/I, denoted $\operatorname{depth}(R/I)$, is the length of the longest maximal sequence contained in $m = (x_1, \ldots, x_n)$. The ***(Krull) dimension*** of R, denoted $\dim R$, is

$$\dim R = \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ is a chain of prime ideals in } R\}.$$

A ring R/I is ***Cohen-Macaulay*** if $\operatorname{depth}(R/I) = \dim(R/I)$. Let M be a graded module over $R = k[x_1, \ldots, x_n]$. We call M ***sequentially Cohen-Macaulay*** if there is a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_s = M$$

of M by graded R-*modules* such that M_i/M_{i-1} is Cohen-Macaulay for all i, and $\dim M_i/M_{i-1} < \dim M_{i+1}/M_i$ for all i, where dim denotes Krull dimension. The above notions can be found in [**6**].

Sequentially Cohen-Macaulay modules were first introduced by Stanley [**22**]. Herzog and Hibi [**17**] gave a classification of sequentially Cohen-Macaulay quotients R/I in terms of the Alexader dual of I, when I is a square-free monomial ideal. This classification extended a criterion of Eagon and Reiner [**5**] for Cohen-Macaulay quotients. Recently many authors have been interested in classifying or identifying (sequentially) Cohen-Macaulay graphs. The graph G is a (sequentially) Cohen-Macaulay graph if $R/I(G)$ is (sequentially) Cohen-Macaulay, where $I(G)$ is the edge ideal of this graph G. J.Herzog and T.Hibi classified Cohen-Macaulay bipartite graphs in [**14**]. S. Faridi showed that all simplicial trees (to be defined below) are sequentially Cohen-Macaulay [**11**]. C.A. Francisco and T. Hà studied how adding "whiskers", a kind of leaf, changes the sequentially Cohen-Macaulay structure. C. A. Francisco and A. Van Tuyl proved that all chordal graphs are sequentially Cohen-Macaulay [**8**]. Recently A. Van Tuyl and R.H. Villarreal complement and extend recent work on this problem by determining when the edge ideal of a bipartite graph is (sequentially) Cohen-Macaulay [**23**].

The main result of this project is:

Theorem 0.1. *(Corollary 4.10) Let $I_t(\Gamma)$ be a path ideal of a tree Γ. Then $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay for all $t \geq 2$.*

To give the proof, we use the notion of a simplicial tree given in [**11**]. Let $\Delta = \langle F_1, \ldots, F_n \rangle$ be a simplicial complex with facets $F_1, \ldots, F_n$. A facet F of Δ is called a ***leaf*** if there exists another facet $H \neq F \in \Delta$ such that $F \cap G \subseteq F \cap H$ for all facet $G \neq F$. A connected simplicial complex Δ is a ***simplicial tree*** if every nonempty subcomplex of Δ has a leaf. We show

Theorem 0.2. *(Theorem 4.9) For all $t \geq 2$ the path ideal $I_t(\Gamma)$ of a tree Γ is the facet ideal of a simplicial tree.*

By applying a result of Faridi that the facet ideals of simplicial trees are sequentially Cohen-Macaulay, Theorem 0.1 then follows.

In Chapter 1 we introduce the basic terminology of graph theory. We also introduce simplicial complexes and simplicial trees. Facet ideals and path ideals are two important ideals to connect a tree and a simplicial complex. They are introduced in Chapter 2. We also introduce the relation between the Stanley-Reisner ideal and facet ideal. At the end of Chapter 2, we give the definition of a graded resolution. In Chapter 3 we introduce Cohen-Macaulay and sequentially Cohen-Macaulay rings, two nice classes of rings. The first three chapters give the basic knowledge for this project. In Chapter 4, the main part of this project, we prove our two main results, Theorem 0.1 and Theorem 0.2. We end with Chapter 5 by describing some future research questions.

CHAPTER 1

Combinatorial Objects

In this chapter we introduce the definitions and notations from combinatorics used throughout this project.

1. Some Graph Theory

DEFINITION 1.1. (***graph, vertex, edge***) [**20**] A ***graph*** is a tuple $G = (V_G, E_G)$ which consists of V_G, a nonempty set of ***vertices*** (or nodes), and E_G, unordered pairs of elements of V_G, called ***edges***. Each edge has either one or two vertices associated with it, called its ***endpoints***. An edge is said to connect its endpoints.

EXAMPLE 1.2. A labeled graph on 5 vertices and 6 edges:

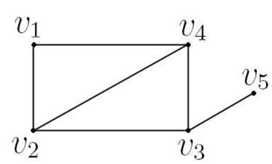

FIGURE 1

Here $V_G = \{v_1, \ldots, v_5\}$ and

$$E_G = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}\}.$$

DEFINITION 1.3. (***loop, multiple edge, multigraph, simple graph***) [**20**] A ***loop*** is an edge whose endpoints are the same. ***Multiple edges*** are two or more edges connecting the same two vertices. The term ***multigraph*** refers to a graph in which multiple edges between nodes are allowed. A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a ***simple graph***. Equivalently, we can say that a simple graph is a graph that contains no loops or multiple edges.

EXAMPLE 1.4. Consider the three graphs below. The first graph is a simple graph because there is no more than one edge between any two vertices and no loops. The second graph is a multigraph because there exist two pairs of vertices connected by two edges in each pair. The third graph is also a multigraph because there are two loops. There is also one pair of vertices connected by two edges.

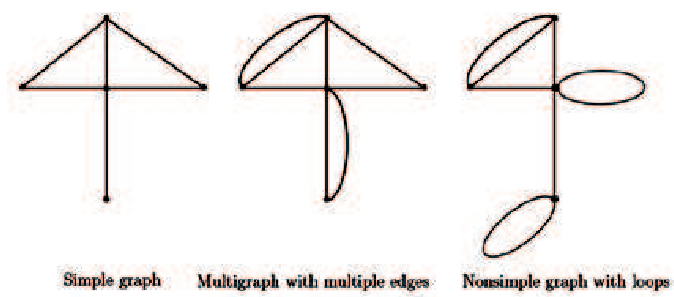

FIGURE 2

DEFINITION 1.5. (***path***, ***length***) A ***path*** in a simple graph is a sequence of vertices $\{v_{i_1}, \ldots, v_{i_l}\}$ such that from each of its vertices there is an edge to the next vertex in the sequence. The ***length*** of a path is the number of edges in this path. The first vertex is called the ***start vertex*** and the last vertex is called the ***end vertex***.

NOTATION 1.6. In order to simplify the expression of a path, we denote $v_{i_1} \cdots v_{i_t}$ as the path of length $t-1$. This means a path starting at v_{i_1} and ending at v_{i_t}. It follows that an edge is a path of length 1 and is denoted by $v_i v_j$, $(i \neq j)$.

EXAMPLE 1.7. Consider the graph of Example 1.2. We say $\{v_1, v_4, v_2, v_3, v_5\}$ is a path of length 4 with start vertex v_1 and end vertex v_5.

DEFINITION 1.8. (***directed graph, directed tree, rooted tree***) [**20**] A ***directed graph*** (or ***digraph***) $G = (V_G, E_G)$ consists of a nonempty set of vertices V_G and a set of ***directed edges*** (or ***arcs***) E_G. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Definition 1.9. (***tree, forest, directed tree, rooted tree***) [**20**] A ***tree*** is a graph in which any two vertices are connected by exactly one path. A ***forest*** is a graph in which any two vertices are connected by at most one path. A ***directed tree*** is a directed graph which would be a tree if the directions on the edges were ignored. A tree is called a ***rooted tree*** if one vertex has been designated, called the ***root***, in which case the edges have a natural orientation, towards or away from the root.

Example 1.10. Let Γ be the directed rooted tree below. The arrows denote the direction.

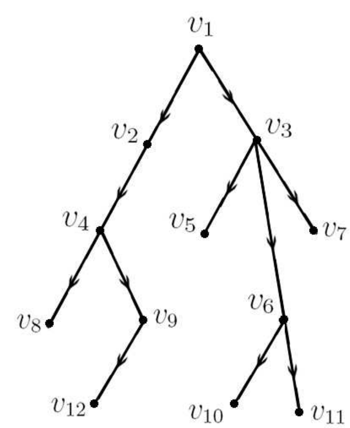

Figure 3

The graph Γ is a tree since there is a unique path between pairs of vertices. Here the vertex v_1 is the root, and the direction of every edge is away from the root. The vertex set of Γ is $V_\Gamma = \{v_1, v_2, \ldots, v_{12}\}$ and the directed edge set of Γ is $E_\Gamma = \{(v_1, v_2), (v_2, v_4), (v_4, v_8), (v_4, v_9), (v_9, v_{12}), (v_1, v_3), (v_3, v_5), (v_3, v_6), (v_3, v_7),$ $(v_6, v_{10}), (v_6, v_{11})\}$.

Definition 1.11. (***degree, leaf***) The ***degree*** of a vertex is the number of edges adjacent to it. We use $\deg x$ to denote the degree of the vertex x. A vertex that has degree 1 is called a ***leaf***.

Example 1.12. Use the graph of Example 1.10. The degrees of the vertices v_8, v_{12}, v_5, v_7, v_{10} and v_{11} are 1, so they are leaves of Γ. We also have $\deg v_1 = \deg v_2 = \deg v_9 = 2$, $\deg v_4 = \deg v_6 = 3$ and $\deg v_3 = 4$. The paths of various lengths are given below:

$$\begin{aligned}
\text{paths of length } 2 &= \{v_1v_2v_4, v_2v_4v_8, v_4v_9v_{12}, v_1v_3v_5, v_1v_3v_6, v_1v_3v_7, v_3v_6v_{10}, v_3v_6v_{11}\} \\
\text{paths of length } 3 &= \{v_1v_2v_4v_8, v_1v_2v_4v_9, v_2v_4v_9v_{12}, v_1v_3v_6v_{10}, v_1v_3v_6v_{11}\} \\
\text{paths of length } t \geq 4 &= \emptyset
\end{aligned}$$

Definition 1.13. (***cycle***) A closed (simple) path, with no repeated vertices other than the starting and ending vertices is a ***cycle***. A ***directed cycle graph*** is a directed version of a cycle, with all the edges being oriented in the same direction. The cycle with n vertices is denoted C_n. The number of vertices in C_n equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it. A cycle is also a path such that the start vertex and end vertex are the same.

Example 1.14. The right-hand graph below is a cycle graph with a cycle $C_6 = \{v_1v_2v_3v_4v_5v_6\}$ in it.

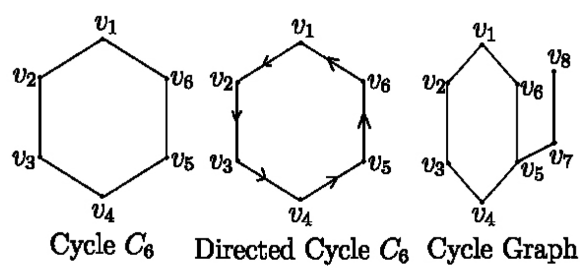

Figure 4

Lemma 1.15. *Let Γ be a connected graph. Then Γ is a tree if and only if Γ has no cycles.*

Proof. Assume there is a cycle $\{v_1, \ldots, v_i, \ldots, v_n, v_1\}$ in the tree Γ. Then there are two different paths connecting v_1 and v_i, that is, $\{v_1, v_2, \ldots, v_i\}$ and $\{v_i, v_{i+1}, \ldots, v_n, v_1\}$. This contradicts Definition 1.9. The converse is the statement that if a connected simple graph is without a cycle, we have there is only one path between pairs of vertices. This means the graph is a tree. □

2. Simplicial complexes

The basic terminology of simplicial complexes is introduced below.

Definition 1.16. (***power set***) If S is a set, then the ***power set*** of S, denoted $P(S)$, is the set of all subsets of S.

Example 1.17. If $S = \{a, b, c\}$, then

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The following definitions can be found in [**22**].

Definition 1.18. (***simplicial complex, face, facet, subcomplex***) Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set. Then an (abstract) ***simplicial complex*** Δ is a subset of the power set $P(V)$ of V, such that:

- $\{v_i\} \in \Delta$ for each $i = 1, \ldots, n$, and
- if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

An element of Δ is a ***face***. The second condition implies $\emptyset$ is a face of Δ. The maximal faces under inclusion are called ***facets***. If $F \in \Delta$, then the dimension of F, is $\dim F = |F| - 1$, where $|F|$ = number of vertices of F. The dimensions of the vertices and edges are 0 and 1, respectively. We set $\dim \emptyset = -1$. We denote the simplicial complex Δ with facets $F_1, \ldots, F_s$ by

$$\Delta = \langle F_1, \ldots, F_s \rangle.$$

The ***subcomplex*** of a simplicial complex Δ is a simplicial complex generated by a subset of the facet set of Δ. That is, $\Delta' = \langle F_{i_1}, \ldots, F_{i_r} \rangle$ is a subcomplex of Δ if $\{F_{i_1}, \ldots, F_{i_r}\} \subset \{F_1, \ldots, F_s\}$.

Definition 1.19. (***pure***) A simpicial complex Δ is ***pure*** if all the facets have the same dimension.

Definition 1.20. ***(dimension of a simplicial complex)*** The ***dimension*** of a simplicial complex Δ is given by $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$.

Example 1.21. Given the simplicial complex $\Delta = \langle \{v_1v_2v_4\}, \{v_1v_3\}, \{v_3v_4\} \rangle$ with facets $F_1 = \{v_1, v_2, v_4\}$, $F_2 = \{v_1, v_3\}$ and $F_3 = \{v_3, v_4\}$, we can draw Δ as

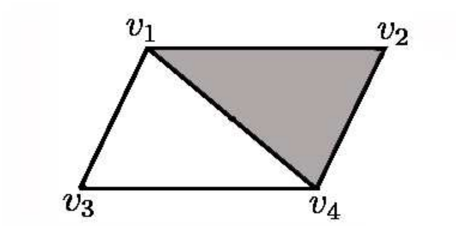

Figure 5

We have $\dim \Delta = 2$, because $\dim F_1 = 2$ and all other facets have dimensions less than or equal to 1.

Definition 1.22. ***(f-vector)*** Let Δ be a simplicial complex of dimension $d-1$. Let f_i denote the number of faces of dimension i in Δ, written as f_i=$f_i(\Delta)$. The ***f-vector*** of Δ is the d-tuple, $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$.

Remark 1.23. It follows directly from the definition that

$$\begin{aligned} f_0 &= |V|, \text{ the number of vertices in } \Delta, \text{ and} \\ f_{-1} &= 1, \text{ because } \emptyset \in \Delta \text{ and } \dim \emptyset = -1. \end{aligned}$$

Example 1.24. Consider the simplicial complex $\Delta = \langle v_1v_2v_3v_4, v_3v_5, v_4v_5, v_1v_5 \rangle$. To draw the picture, we know the facet $\{v_1, v_2, v_3, v_4\}$ is a tetrahedron. For this tetrahedron $\{v_1, v_2, v_3, v_4\}$, all triangles $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$, the edges of the triangles $\{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_5\}$ and vertices $v_1, \ldots, v_5$ are included in this facet. The other three facets are three edges: $\{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_5\}$. Then Δ looks like Figure 6. So there is one tetrahedron, four triangles, nine edges and five vertices in this simplicial complex. Then the f-vector is $f(\Delta) = (5, 9, 4, 1)$.

Definition 1.25. [**11**] (***leaf of a simplicial complex***) Let $\Delta = \langle F_1, \ldots, F_n \rangle$ be a simplicial complex with facets $F_1, \ldots, F_n$. A facet F of Δ is called a ***leaf*** if it is the only facet or if there exists another facet $H \neq F \in \Delta$ such that $F \cap G \subseteq F \cap H$ for all facets $G \neq F$.

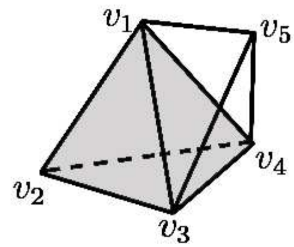

FIGURE 6

EXAMPLE 1.26. Consider the simplicial complex with facets $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_3, x_4, x_5\}$, and $F_3 = \{x_4, x_5, x_6\}$ (see Figure 7). Then F_1 and F_3 are leaves since we can use $H = F_2$.

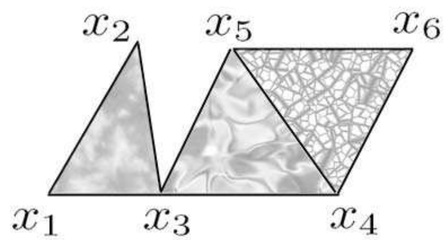

FIGURE 7

Indeed,

$$\begin{aligned} F_1 \cap F_2 &= \{x_3\} \subseteq F_1 \cap H \\ F_1 \cap F_3 &= \emptyset \subseteq F_1 \cap H \\ F_3 \cap F_1 &= \emptyset \subseteq F_3 \cap H \\ F_3 \cap F_2 &= \{x_4, x_5\} \subseteq F_3 \cap H. \end{aligned}$$

However, F_2 is not a leaf since there is no facet H such that

$$\begin{aligned} F_2 \cap F_1 &= \{x_3\} \subseteq F_2 \cap H \textit{ and} \\ F_2 \cap F_3 &= \{x_4, x_5\} \subseteq F_2 \cap H. \end{aligned}$$

DEFINITION 1.27. ***(free vertex)*** In a simplicial complex, a vertex is a ***free*** vertex if it belongs to only one facet.

LEMMA 1.28. *If F is a leaf in a simplicial complex Δ, then F has a free vertex.*

PROOF. [**22**] Suppose the facet F is a leaf in Δ. By definition, there is another facet $H \in \Delta$ such that $F \cap G \subseteq F \cap H$, where $G \neq F$ and $H \neq F$. So there must exist a vertex $x \in F$, but $x \notin H$. It follows $x \notin (F \cap H)$ and $x \notin (G \cap F) \subseteq (H \cap F)$. Thus this vertex x is only in F. □

EXAMPLE 1.29. The converse statement of Lemma 1.28 is not true, as shown in this example. Consider the simplicial complex $\Delta = \langle F_1, F_2, F_3 \rangle$, where $F_1 = \{v_1, v_2, v_3\}$, $F_2 = \{v_2, v_4, v_5\}$ and $F_3 = \{v_3, v_5, v_6\}$.

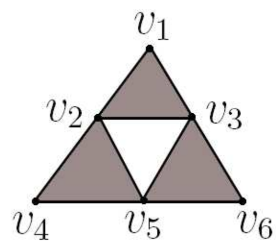

FIGURE 8

Every facet has a free vertex but no facet is a leaf of Δ. For example, the vertex v_1 is free in facet F_1, and we know $F_2 \cap F_1 = \{v_2\}$ and $F_3 \cap F_1 = \{v_3\}$. But we can not find a facet containing these two vertices together.

DEFINITION 1.30. ***(Simplicial tree, simplicial forest)*** A connected simplicial complex Δ is a ***simplicial tree*** if every nonempty subcomplex of Δ has a leaf. A simplicial complex is a ***simplicial forest*** if every nonempty subcomplex of Δ has a leaf.

EXAMPLE 1.31. The simplicial complex in Example 1.26 is a simplicial tree. The simplicial complex $\Delta = \langle v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4, v_3v_5v_6 \rangle$ is not a simplicial tree (see Figure 9). Though facet $F = \{v_3, v_5, v_6\}$ is a leaf and has free vertices v_5 and v_6, any facet in the subcomplex $\Delta' = \langle v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4 \rangle$ is not a leaf, i.e., for any facet $F \in \Delta'$, there is no facet $H \in \Delta$ such that $F \cap G \subseteq F \cap H$ for all facets $G \neq F$.

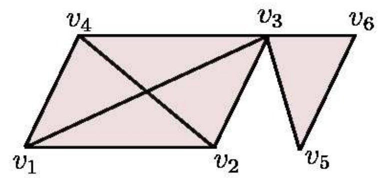

FIGURE 9

CHAPTER 2

Monomial Ideals and Resolutions

Now that we have some basic knowledge in *graph theory*, in this chapter we will introduce the researched objects in this paper, the edge ideal and the path ideal. We first give the definition of a square-free monomial ideal in a polynomial ring. The edge ideal, the path ideal, the facet ideal and the Stanley-Reisner ideal introduced below are all examples of square-free monomial ideals. In Chapter 1, we use v_i to denote a vertex in a graph. If we regard a vertex as a variable in the ring $R = k[x_1, \ldots, x_n]$, we can also use x_i to denote a vertex.

1. Square-free mononial ideals

DEFINITION 2.1. (***monomial, square-free monomial ideal***) Let $k[x_1, \ldots, x_n]$ denote the ***polynomial ring in the variables*** $x_1, \ldots, x_n$ with coefficients in field k. A ***monomial*** in $k[x_1, \ldots, x_n]$ is a term of the form $M = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. We say M is ***square-free*** if $0 \leq a_i \leq 1$ for all i. An ideal I is a ***(square-free) monomial ideal*** if I is generated by (square-free) monomials.

EXAMPLE 2.2. $I = (x_3x_4, x_1x_2x_3)$ is a square-free monomial ideal in $k[x_1, x_2, x_3, x_4]$.

DEFINITION 2.3. (***Stanley-Reisner ideal, Stanley-Reisner ring***) If Δ is a simplicial complex on $V_\Delta = \{x_1, \cdots, x_n\}$, then the ***Stanley-Reisner ideal*** is

$$I_\Delta = (x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \cdots, x_{i_r}\} \notin \Delta) \subseteq k[x_1, \ldots, x_n].$$

The generators of I_Δ correspond to the nonfaces of Δ. The ***Stanley-Reisner ring*** is the quotient ring

$$k[x_1, \ldots, x_n]/I_\Delta.$$

EXAMPLE 2.4. Consider the simplicial complex $\Delta = \langle \{x_1, x_2, x_4\}, \{x_1, x_3\}, \{x_3, x_4\} \rangle$. Then the nonfaces are

$$\{x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}.$$

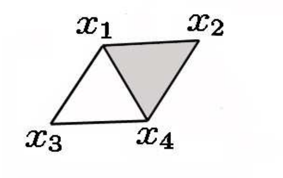

Figure 1

Thus, the Stanley-Reisner ideal I_Δ is given by

$$I_\Delta = (x_2x_3, x_1x_3x_4, x_1x_2x_3, x_2x_3x_4, x_1x_2x_3x_4).$$

Note that our set of generators is not a minimal set of generators. For example, since $x_2x_3 \in I_\Delta$, it follows from the definition of an ideal that $x_1x_2x_3x_4 = (x_1x_4)(x_2x_3) \in I_\Delta$. Thus, if remove the generator $x_1x_2x_3x_4$ from the above list, the remaining generators still generate I_Δ. More precisely, I_Δ is generated by the minimal nonfaces of Δ. So, in our example,

$$I_\Delta = (x_2x_3, x_1x_3x_4).$$

Remark 2.5. I_Δ is always a square-free monomial ideal. In fact, we have a bijection:

$$\{\text{simplicial complexes}\} \longleftrightarrow \{\text{square-free monomial ideals}\}$$

given by $\Delta \longmapsto I_\Delta$.

We introduce the facet ideal, a monomial ideal associated to a simplicial complex. Facet ideals were first defined by Sara Faridi [**10**].

Definition 2.6. ***(facet ideal, facet complex)*** [**10**]

- Let Δ be a simplicial complex over $V = \{x_1, \ldots, x_n\}$. The set V corresponds to the n variables $x_1, \ldots, x_n$ in a polynomial ring $R = k[x_1, \ldots, x_n]$ with k a field. The ***facet ideal*** of Δ, denoted $\mathcal{I}(\Delta)$, is the ideal generated by all square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_t}$, where $\{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ is a facet of Δ. Thus $\mathcal{I}(\Delta) = (x_{i_1}\cdots x_{i_t} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\} \text{ is a facet of } \Delta)$.
- Let $I = (M_1, \ldots, M_q)$ be an ideal of R, where $M_1, \ldots, M_q$ are square-free monomials in $x_1, \ldots, x_n$ that form a minimal set of generators for I. We define the ***facet complex*** of I, denoted by $\delta_{\mathcal{F}}(I)$ to be the simplicial

complex over a set of vertices $x_1, \ldots, x_n$ with facets $F_1, \ldots, F_q$, where for each i, $F_i = \{x_j \mid x_j | M_i, 1 \leq j \leq n\}$.

Example 2.7. Let $\Delta = \langle xyz, uvz, yu \rangle$.

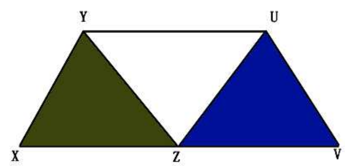

Figure 2

Then $\mathcal{I}(\Delta) = (xyz, uvz, yu)$ is the facet ideal of Δ in $R = k[x, y, z, u, v]$.

Facet ideals also give a one-to-one correspondence between simplicial complexes and square-free monomial ideals, i.e.

$$\{\text{simplicial complexes}\} \longleftrightarrow \{\text{square-free monomial ideals}\}$$

$$\Delta \longmapsto \mathcal{I}(\Delta)$$

Example 2.8. We show two ways one can associate to a square-free monomial ideal a simplicial complex. Let $I = (x_1x_3, x_1x_4, x_1x_5, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5)$ be the Stanley-Reisner ideal (nonface ideal) of a simplicial complex Δ_1. Then the generators correspond to the minimal nonfaces. So, the square-free monomials of $R = k[x_1, x_2, x_3, x_4, x_5]$ that are not in I correspond to the faces of the simplicial complex Δ_1. For example, x_1x_2 is not in I, so there is an edge between x_1 and x_2. Through the Stanley-Reisner correspondence we can find all the facets and get the simplicial complex Δ_1 (see Figure 3).

Let $I = (x_1x_3, x_1x_4, x_1x_5, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5)$ be the facet ideal of another simplicial complex Δ_2, i.e. $I = \mathcal{I}(\Delta_2)$. The generators are all the facets of Δ_2 and this simplicial complex looks like Figure 4.

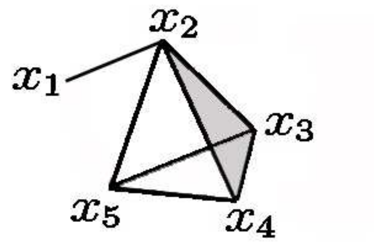

FIGURE 3

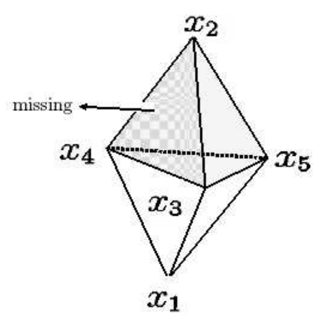

FIGURE 4

2. Relationships between nonface ideal and facet ideal

In this section we will build up the relationships between the nonface ideal (Stanley-Reisner ideal) and the facet ideal by using the cover dual and the Alexander dual.

DEFINITION 2.9. (***Square-free Alexander dual***) Let I be a square-free monomial ideal in $R = k[x_1, x_2, \ldots, x_n]$. The ***square-free Alexander dual*** of

$$I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t})$$

is the ideal

$$I^{\vee} = (x_{1,1}, \ldots, x_{1,s_1}) \cap \cdots \cap (x_{t,1}, \ldots x_{t,s_t})$$

Example 2.10. Let $\mathcal{I}(\Delta)$ be the facet ideal of the simplicial complex Δ in Figure 5. So

$$\mathcal{I}(\Delta) = (x_1x_2x_3, x_1x_3x_4, x_1x_2x_5, x_1x_4x_5).$$

Then

$$\begin{aligned} \mathcal{I}(\Delta)^\vee &= (x_1, x_2, x_3) \cap (x_1, x_3, x_4) \cap (x_1, x_2, x_5) \cap (x_1, x_4, x_5) \\ &= (x_2x_4, x_1, x_3x_5). \end{aligned}$$

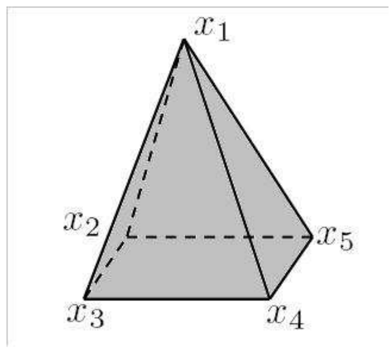

Figure 5

Definition 2.11. (***minimal vertex cover***) A vertex cover for Δ is a subset P of V that intersects every facet of Δ, i.e. $P \cap F_i \neq \emptyset$, where F_i is any facet of Δ. If P is a minimal element of the set of vertex covers of Δ, then P is called a ***minimal vertex cover***.

Example 2.12. Let Δ be the simplicial complex as in the above Example 2.7. Then the vertex covers of Δ are $\{y, u\}, \{y, z\}, \{u, z\}, \{x, u\}, \{y, v\}, \{x, y, u\}, \{v, y, u\}, \ldots$. So $\{y, u\}, \{y, z\}, \{u, z\}, \{x, u\}, \{y, v\}$ are the minimal vertex covers of Δ.

Definition 2.13. (***cover complex***) Let Δ_M be the simplicial complex whose facets are the minimal vertex covers of Δ. We call Δ_M the ***cover complex***.

Example 2.14. If $\Delta = \langle xyz, uvz, yu \rangle$ is the simplicial complex above in Figure 6, then the cover complex of Δ is $\Delta_M = \langle yu, yz, uz, xu, yv \rangle$. This simplicial complex looks like Figure 7.

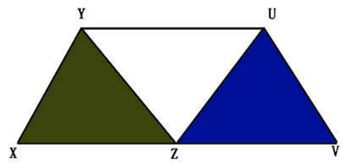

FIGURE 6

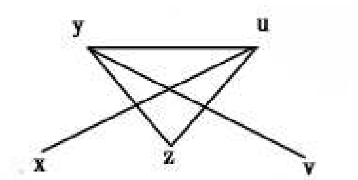

FIGURE 7

THEOREM 2.15. *If Δ is a simplicial complex, then Δ_M is a dual of Δ; i.e. $(\Delta_M)_M = \Delta$.*

See the proof of Proposition 10 in [**11**].

EXAMPLE 2.16. Let $I = (xyz, uvz, yu)$. Then this is the facet ideal of Figure 6, but it's also the Stanley-Reisner ideal of the simplicial complex in Figure 8. The faces xyz and zuv are missing in the simplicial complex of Figure 8.

DEFINITION 2.17. Let Δ be a simplicial complex. Then the ***Alexander dual*** of Δ is the simplicial complex

$$\Delta^\vee = \{F \subset V \ \mid \ F^c \notin \Delta\}, \text{where } F^c = V - F.$$

We also have $\Delta^{\vee\vee} = \Delta$.

LEMMA 2.18. *If Δ is a simplicial complex and $\Delta^\vee$ is the Alexander dual of Δ, then $I_{\Delta^\vee} = I_\Delta^\vee$.*

The following examples will show the relationships between the facet ideal and the nonface ideal step by step.

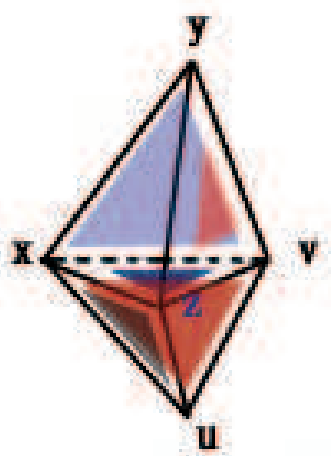

FIGURE 8

EXAMPLE 2.19. Let $I = (xyz, uvz, yu)$. Then I is the facet ideal of the simplicial complex in Figure 5. Then the facet ideal of Δ_M (Δ_M is the cover complex of Δ) is

$$J = \mathcal{I}(\Delta_M) = (yu, yz, uz, xu, yv).$$

Let Δ be the simplicial complex associated to I via the Stanley-Reisner correspondence. This is the simplicial complex in the previous Example 2.7. Then the Alexander dual of Δ is $\Delta^\vee = \langle xy, xvz, uv\rangle$. It looks like Figure 9.

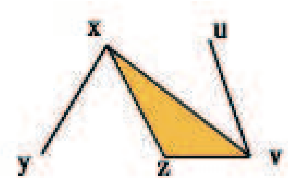

FIGURE 9

The ideal J is also the nonface ideal of $\Delta^\vee$, since the nonface ideal of $\Delta^\vee$ is (yu, yz, uz, xu, yv). So the facet ideal of Δ_M, i.e. $\mathcal{I}(\Delta_M)$, is equal to the nonface ideal of $\Delta^\vee$. That is $\mathcal{I}(\Delta_M) = I_\Delta^\vee$.

Example 2.20. The following picture (Figure 10) summarizes the relationships among the facet ideal, the Stanley-Reisner ideal, cover dual and Alexander dual of a simplicial complex.

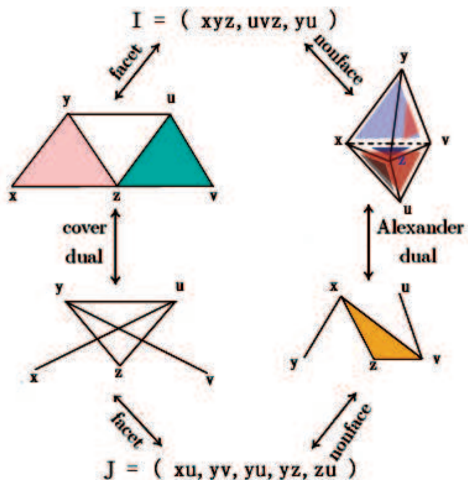

Figure 10

3. Edge ideals and path ideals

In this section, we introduce a special class of monomial ideals, edge ideals and path ideals. Edge ideals were introduced by Villarreal in [**24**]. To generalize the definition of an edge ideal, Conca and De Negri first introduced the definition of a path ideal in [**3**]. Edge ideals and path ideals are examples of facet ideals. They are the simplest type of facet ideals. They are generated by square-free monomials of degree two or higher and they can be associated to graphs or to complexes via the Stanley-Reisner correspondence.

Definition 2.21. (***edge ideals***) The ***edge ideal*** $I(G)$ associated to the graph G is the ideal of R generated by the set of square-free monomials x_ix_j such that x_i and x_j are adjacent, that is, $\{x_i, x_j\}$ is an edge of G. Hence

$$I(G) := (\{x_ix_j \mid \{x_i, x_j\} \in E(G)\}).$$

Definition 2.22. (***path ideal***) The ***path ideal*** $I_t(G)$ associated to the graph G is the ideal of R generated by the set of square-free monomials $x_{i_1} \cdots x_{i_t}$ such that the sequence $\{v_{i_1}, \dots, v_{i_t}\}$ is connected one by one, that is $\{v_{i_1}, \dots, v_{i_t}\}$ is a path in G.

REMARK 2.23. When $t = 2$, $I_2(G) = I(G)$.

EXAMPLE 2.24. Given a cycle graph as in Figure 11, the root is x_1. We can find the path ideal generated by all paths of length 2 (i.e. there are 3 vertices in every path)

$$I_3(\Gamma) = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2).$$

And the edge ideal is

$$I(\Gamma) = I_2(\Gamma) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1).$$

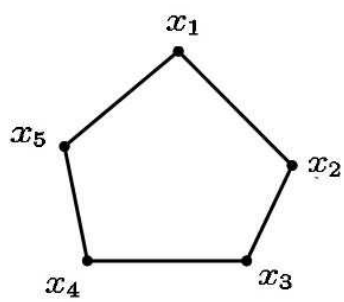

FIGURE 11

4. Hilbert Series.

The Stanley-Reisner ring R/I_Δ encodes information about the simplicial complex Δ. In the section below we show how some of the information is encoded and provide background knowledge in order to introduce the definition of graded resolutions.

DEFINITION 2.25. (***homogeneous***) A polynomial $F \in R = k[x_1, \dots, x_n]$ is ***homogeneous*** if all its terms have the same degree.

EXAMPLE 2.26. The following are examples in $R = k[x_1, x_2, x_3]$:

- $3x_1x_2x_3 + 4x_1^2x_2 + 7x_3^3$ ←homogeneous.

- $3x_1x_2x_3 + 4x_3^7 \leftarrow$ not homogeneous.
- a monomial is always homogenous.

Set $R_i = \{F \in R \mid F \text{ homogeous with} \deg F = i\}$. Then R is a graded ring, i.e.

$$R = \bigoplus_{i \in \mathbb{N}} R_i \text{ and } R_i R_j \subseteq R_{i+j}.$$

The set R_i is a vector space over k. A basis for R_i is the set of all monomials of degree i.

Example 2.27. If $R = k[x_1, x_2]$, R_3 is the k-vector space with basis $B = \{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$.

Lemma 2.28. *If* $R = k[x_1, \dots x_n]$, *then* $\dim_k R_i = \binom{n-1+i}{i}$.

Proof. The set R_i consists of all homogeneous elements of R of degree i. A basis for R_i is the set of all monomials of the form $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 + \cdots + a_n = i$. Count integer solutions to this equation and the lemma holds. $\square$

Definition 2.29. An ideal I is ***homogenous*** if I is generated by homogenous elements.

Let $I_i = I \cap R_i$. This is the set of all homogenous elements of degree i in I. Furthermore, I_i is a subspace of R_i. If I is a homogenous ideal of $R = k[x_1, \dots, x_n]$, then R/I is also a graded ring. That is

$$R/I = \bigoplus_{i \in \mathbb{N}} (R/I)_i.$$

Here $(R/I)_i$ is the k-vector space R_i/I_i. We then have

$$\dim_k (R/I)_i = \dim_k R_i - \dim_k I_i.$$

We can encode the information about the dimensions into a generating function.

Definition 2.30. (***Hilbert series***) Let I be a homogenous ideal of $R = k[x_1, \dots, x_n]$. The ***Hilbert series of*** R/I is the formal power series.

$$HS(R/I, t) = \sum_{i \in \mathbb{N}} (\dim_k (R/I)_i)\, t^i.$$

EXAMPLE 2.31. Suppose $R = k[x_1, x_2]$ and $I = (0)$. So $R/I = R$ and $\dim_k(R/I)_i = \dim_k R_i$. Since a basis for R_i is given by $\{x_1^i, x_1^{i-1}x_2, \ldots, x_1x_2^{i-1}, x_2^i\}$, we have $\dim_k R_i = (i+1)$. So

$$HS(R,t) = \sum_{i\in\mathbb{N}} (i+1)t^i = 1 + 2t + 3t^2 + 4t^3 + \cdots$$

But $1/(1-t)^2 = 1 + 2t + 3t^2 + 4t^3 + \cdots$, so

$$HS(R,t) = 1/(1-t)^2.$$

When I is a homogenous ideal of $R = k[x_1, \ldots, x_n]$, the Hilbert series $HS(R/I, t)$ is always a rational function.

THEOREM 2.32. [**19**, Corollary 1.15] *Suppose R/I has dimension d. Then there exists a unique polynomial $h(t) = h_0 + h_1t + \cdots + h_lt^l \in \mathbb{Z}[t]$ such that $h(1) \neq 0$ and*

$$HS(R/I, t) = \frac{h_0 + h_1t + \cdots + h_lt^l}{(1-t)^d}.$$

DEFINITION 2.33. (***h-vector***) Suppose the Hilbert series of R/I is

$$HS(R/I, t) = \frac{h_0 + h_1t + \cdots + h_lt^l}{(1-t)^d}.$$

Then the ***h-vector*** of R/I is the tuple

$$h(R/I) = (h_0, h_1, \ldots, h_l).$$

The f-vector (as introduced in Definition 2.33) can be used to compute the Hilbert series of R/I_Δ.

THEOREM 2.34. [**19**, Corollary 1.15] *Let Δ be a simplicial complex with f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$. Then*

$$HS(R/I_\Delta, t) = \sum_{i=-1}^{d-1} \frac{f_it^{i+1}}{(1-t)^{i+1}}.$$

EXAMPLE 2.35. Consider the simplicial complex Δ as in Example 2.4 and given in Figure 12:

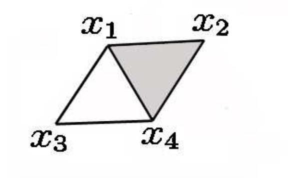

FIGURE 12

The f-vector of Δ is $f(\Delta) = (4, 5, 1)$. So, the Hilbert series of R/I_Δ is given by

$$\begin{aligned} HS(R/I_\Delta, t) &= \frac{f_{-1}t^0}{(1-t)^0} + \frac{f_0 t^1}{(1-t)} + \frac{f_1 t^2}{(1-t)^2} + \frac{f_2 t^3}{(1-t)^3} \\ &= 1 + \frac{4t}{(1-t)} + \frac{5t^2}{(1-t)^2} + \frac{1t^3}{(1-t)^3} \\ &= \frac{(1-t)^3 + 4t(1-t)^2 + 5t^2(1-t) + t^3}{(1-t)^3} \\ &= \frac{1+t+t^3}{(1-t)^3}. \end{aligned}$$

The h-vector of Δ is then given by $h(\Delta) = (1, 1, 0, 1)$.

There is a relationship between the f-vectors and h-vectors:

LEMMA 2.36 ([**22**]).

$$h_j = \sum_{i=0}^{j} (-1)^{j-1} \binom{d-i}{j-i} f_{i-1} \textit{ and } f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$

EXAMPLE 2.37. This example will use a method similar to Pascal's triangle to compute the h-vector when given the f-vector. Consider the simplicial complex $\Delta = \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_2x_5, x_3x_5 \rangle$. Since the f-vector is $f(\Delta) = (5, 8, 4)$, we can find the h-vector using a "Pascel triangle like" method:

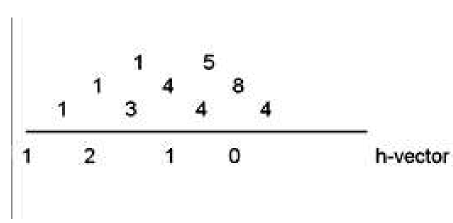

FIGURE 13

In this triangle, the numbers on the left side are all one's and those on the right side are the numbers of the f-vector. Every other number in this triangle equals the subtraction of the two numbers above it.

5. Building a minimal resolution

We will build a minimal free resolution of an ideal I in the ring $R = k[x_1, \ldots, x_n]$ in this section.

Let $R = k[x_1, \ldots, x_n]$ where k is a field, and let

$$R^n = \left\{ \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \;\middle|\; f_i \in R = k[x_1, \ldots, x_n] \right\}.$$

Note that R^n is a free R-module under the operation

$$g \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} gf_1 \\ \vdots \\ gf_n \end{bmatrix} \text{ with } g \in R \text{ and } \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in R^n.$$

Definition 2.38. (R***-module homomorphism***) A function $T : R^n \to R^m$ is an R***-module homomorphism*** if $T(\underline{x} + \underline{y}) = T(\underline{x}) + (\underline{y})$ for all $\underline{x}, \underline{y} \in R^n$ and $T(c\underline{x}) = cT(\underline{x})$ for all $c \in R$ and $\underline{x} \in R^n$.

Theorem 2.39. *If $T : R^n \to R^m$ is an R-module homomorphism, then there exists an $m \times n$ matrix with entries in R such that $T(\underline{x}) = A\underline{x}$. In particular,*

$$A = \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & \cdots & T(\underline{e}_n) \end{bmatrix}$$

where $\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in R^n.$

Proof. Write $\underline{x} = I_n\underline{x} = [\underline{e}_1 \cdots \underline{e}_n]x = x_1\underline{e}_1 + \cdots + x_n\underline{e}_n$, and use the linearity of T to compute

$$\begin{aligned} T(\underline{x}) &= T(x_1e_1 + \cdots + x_ne_n) = x_1T(\underline{e}_1) + \cdots + x_nT(\underline{e}_n) \\ &= [T(\underline{e}_1)\cdots T(\underline{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\underline{x}. \end{aligned}$$

$\square$

Example 2.40. If $R = k[x, y, z]$ and $T : R^3 \to R^3$ is the R-module homomorphism given by

$$T\left(\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right) = \begin{bmatrix} f_1x + f_2z \\ -f_1x + f_3z \\ -f_2x - f_3y \end{bmatrix}$$

then

$$T\left(\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right) = \begin{bmatrix} x & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Every R-module has a free resolution, and if R is graded, every graded R-module has a graded free resolution. To construct such a resolution, begin by taking a set of minimal generators for an ideal I and map a free module onto I by mapping the generators of the free module to the given generators of I. For example, suppose $I = (F_{0,1}, \ldots, F_{0,t_0})$ is an ideal of R. We can construct an R-module homomorphism

$$\varphi_0 : R^{t_0} \to I \subseteq R^1$$

by

$$\varphi_0\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix}\right) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0} = \begin{bmatrix} F_{0,1} & F_{0,2} & \cdots & F_{0,t_0} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_{t_0} \end{bmatrix}.$$

Definition 2.41. (***syzygy***) Let M be an R-module and suppose $\{F_1, \ldots, F_t\} \subseteq M$. A ***syzygy*** of $F_1, \ldots, F_t$ is a t-tuple $\begin{bmatrix} G_1 \\ \vdots \\ G_t \end{bmatrix} \in R^t$ such that

$$G_1F_1 + \cdots + G_tF_t = 0.$$

We make some observations about the map φ_0:

(1)

$$\begin{aligned} \ker\varphi_0 &= \left\{ \begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix} \middle| \varphi_0\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix}\right) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0} = 0 \right\} \\ &= \{\text{all syzygies of } F_{0,1}, \ldots, F_{0,t_0}\} \end{aligned}$$

We call $\ker\varphi_0$ the ***first syzygy module*** of I.

(2) $\ker\varphi_0$ is a finitely generated submodule of R^{t_0}, i.e., there exists $\underline{F}_{1,1}, \ldots, \underline{F}_{1,t_1}$, such that

$$\ker\varphi_0 = \langle \underline{F}_{1,1}, \ldots, \underline{F}_{1,t_1} \rangle = \{G_1\underline{F}_{1,1} + \cdots + G_{t_1}\underline{F}_{1,t_1} \mid G_i \in R\}.$$

(3) $\ker\varphi_0$ is like the null space of matrix A, i.e.

$$\mathrm{Nul}(A) = \{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} = \overrightarrow{0}\}.$$

We can now define a map $\varphi_1 : R^{t_1} \to \ker\varphi_0 \subseteq R^{t_0}$ by

$$\varphi_1\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_1} \end{bmatrix}\right) = G_1\underline{F}_{1,t_1} + \cdots + G_{t_1}\underline{F}_{1,t_1} = \begin{bmatrix} \underline{F}_{1,1} & \underline{F}_{1,2} & \cdots & \underline{F}_{1,t_1} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_{t_1} \end{bmatrix}$$

We make some further observations:

(1) $\ker\varphi_1$ is called the second syzygy module.

(2) $\ker\varphi_1$ measures the relations among the generators of $\ker\varphi_0$.

(3) $\ker\varphi_1$ is a finitely generated R module, i.e., there exist $\underline{F}_{2,1}, \ldots, \underline{F}_{2,t_2} \in \ker\varphi_1$ such that

$$\ker\varphi_1 = \langle \underline{F}_{2,1}, \ldots, \underline{F}_{2,t_2} \rangle$$

We can repeat the above step to now create a map $\varphi_2 : R^{t_2} \to \ker\varphi_1 \subseteq R^{t_1}$. In fact, we continue to reiterate this process. Eventually, this process will stop because of the following theorem:

THEOREM 2.42 (Hilbert Syzygy Theorem). [**6**, Theorem 19.7] *If $R = k[x_1, \ldots, x_n]$, then there exists an $l \leq n$ such that $\ker\varphi_l = 0$, i.e., the l^{th} syzygy module is 0.*

We now tie the above ideas together to describe the resolution of an ideal. Associated to any ideal $I \subseteq R = k[x_1, \ldots, x_n]$ is a minimal free resolution of the form

$$0 \longrightarrow R^{t_l} \xrightarrow{\varphi_l} R^{t_{l-1}} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_2} R^{t_1} \xrightarrow{\varphi_1} R^{t_0} \xrightarrow{\varphi_0} I \longrightarrow 0$$

where

- $l \leq n$,
- $\operatorname{Im}\varphi_{i+1} = \ker\varphi_i$, and
- each φ_i is represented by a $t_{i-1} \times t_i$ matrix with entries in R.

DEFINITION 2.43. (i^{th} ***Betti number***) The i***th Betti number of*** I, denoted $\beta_i(I)$, equals t_i, the rank of R appearing in the ith step of the resolution. The number $\beta_i(I)$ is the number of minimal generators of $\ker\varphi_{i-1}$.

REMARK 2.44. The $0th$ Betti number $\beta_0(I)$ equals the number of minimal generators of I.

EXAMPLE 2.45. Let $R = k[x, y, z]$ and $I = (x^2, y^2, z)$. Then the minimal resolution is

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} z \\ -y^2 \\ x^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y^2 & z & 0 \\ -x^2 & 0 & z \\ 0 & -x^2 & -y^2 \end{bmatrix}} R^3 \xrightarrow{[x^2 \; y^2 \; z]} I \longrightarrow 0.$$

So $\beta_0(I) = 3$, $\beta_1(I) = 3$ and $\beta_2(I) = 1$.

6. The graded resolution

Return to the example $I = (x^2, y^2, z)$ in $R = k[x, y, z]$. The elements in each matrix defining a map are in fact homogeneous elements. The degrees of these elements are also of interest. We can modify the construction so that we can extract this information.

DEFINITION 2.46. Let M and N be graded R-modules, i.e.

$$M = \bigoplus_{i\in\mathbb{Z}} M_i \text{ and } N = \bigoplus_{i\in\mathbb{Z}} N_i$$

An R-module ***homomorphism*** $\varphi : M \longrightarrow N$ ***is graded of degree 0*** if $\varphi(M_a) \subseteq N_a$ for all a, i.e., degree a elements of M are mapped to elements of degree a of N.

DEFINITION 2.47. If $R = k[x_1, \ldots, x_n]$, then the graded R-module shifted by $a \in \mathbb{N}$ is $R(-a)$ where

$$R(-a)_i = R_{i-a},$$

that is, the degree i part of $R(-a)$ equals the degree $i - a$ part of R.

EXAMPLE 2.48. $1 \in R(-5)$ has $\deg 1 = 5$, since $1 \in R(-5)_5 = R_{5-5} = R_0$. Similarly, $x^2 + y^2 \in R(-5)$ has $\deg x^2 + y^2 = 7$, since $x^2 + y^2 \in R(-5)_7 = R_2$.

DEFINITION 2.49. (***homogeneous of degree*** d) Let $d_1, \ldots, d_t \in \mathbb{N}$. Then

$$R(-d_1) \oplus \ldots \oplus R(-d_t) = \{(G_1, \ldots, G_t) \ \mid \ G_i \in R(-d_i)\}.$$

We say $(G_1, \ldots, G_t)$ is ***homogeneous of degree*** d in $R(-d_1) \oplus \ldots \oplus R(-d_t)$ if $G_i \in R(-d_i)_d$ for each i.

EXAMPLE 2.50. $(x^2 + y^2, zxy) \in R(-5) \oplus R(-4)$ is homogeneous of degree 7.

Let $I = (F_{0,1}, \ldots, F_{0,t_0})$ be a homogeneous ideal with degree $\deg F_{0,i} = d_{0,i}$. Define a map

$$\varphi_0 : R(-d_{0,1}) \oplus R(-d_{0,2}) \oplus \cdots \oplus R(-d_{0,t_0}) \longrightarrow (F_{0,1}, \ldots, F_{0,t_0}) \subseteq R$$

by

$$\varphi_0((G_1, \ldots, G_{t_0})) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0}.$$

Then the map φ_0 has degree 0. To see this, note that if $(G_1, \ldots, G_{t_0})$ is homogeneous of degree d in $R(-d_{0,1}) \oplus \ldots \oplus R(-d_{0,t_0})$ then $\deg G_i = d - d_{0,i}$ in R. So

$$\varphi((G_1, \ldots, G_{t_0})) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0} \text{ is homogeneous of degree } d.$$

One can show that $\ker \varphi_0 = \langle \underline{F}_{1,1}, \ldots, \underline{F}_{1,t_1} \rangle$ is generated by homogeneous elements in $R(-d_{0,1}) \oplus \cdots \oplus R(-d_{0,t_0})$ of degree $d_{1,1}, \ldots, d_{1,t_1}$, respectively. Repeat the above idea to get a map

$$\varphi_1 : R(-d_{1,1}) \oplus \cdots \oplus R(-d_{1,t_1}) \to \ker \varphi_1 \subseteq R(-d_{0,1}) \oplus \cdots \oplus R(-d_{0,t_0})$$

defined by

$$(G_1, \ldots, G_t) \mapsto \begin{bmatrix} \underline{F}_{1,1} & \ldots & \underline{F}_{1,t_1} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_t \end{bmatrix}.$$

Again, $\ker \varphi_1$ is generated by homogeneous elements. We continue to reiterate this process until $\ker \varphi_l = 0$ for some l (which is guaranteed by the Hilbert Syzygy Theorem. See Theorem 2.42).

So, associated to any homogeneous ideal $I \subseteq R = k[x_1, \ldots, x_n]$ is a minimal graded free resolution of the form

$$0 \longrightarrow \mathcal{F}_l \xrightarrow{\varphi_l} \mathcal{F}_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} I \longrightarrow 0$$

where

- $l \leq n$
- φ_i is a matrix with homogeneous entries in R
- $\mathcal{F}_i = R(-d_{i,1}) \oplus \cdots \oplus R(-d_{i,t_i})$.

Example 2.51. Now we answer the problem mentioned at the beginning of this section. We give the minimal graded free resolution of Example 2.45. The minimal graded free resolution of I is

$$0 \longrightarrow R(-5) \longrightarrow R^2(-3) \oplus R(-4) \longrightarrow R(-1) \oplus R^2(-2) \longrightarrow I \longrightarrow 0.$$

Definition 2.52. (***projective dimension***) The minimum length of a free resolution is called the ***projective dimension*** of I over R, written $pd(I)$ for short.

Proof. Special case of [**6**, Theorem 19.9], when $M = R/I$. □

Definition 2.53. (i, j^{th} ***graded Betti number*** i, j) The i, j^{th} ***graded Betti number*** I, denoted $\beta_{i,j}(I)$, equals the number of times $R(-j)$ appears in $\mathcal{F}_i$. Equivalently, $\beta_{i,j}(I)$ is the number of minimal generators of degree j of $\ker \varphi_{i-1}$.

Example 2.54. Given the tree Γ in Figure 14, we consider the path ideal $I_4(\Gamma) = (x_1x_2x_4x_8, x_1x_2x_4x_9, x_2x_4x_9x_{12}, x_1x_3x_6x_{10}, x_1x_3x_6x_{11}) \subseteq R = k[x_1, \ldots, x_{12}]$. The minimal free resolution of $I_4(\Gamma)$:

$$0 \longrightarrow R(-9) \longrightarrow R^4(-8) \longrightarrow R^3(-5) \oplus R^4(-7) \longrightarrow R^5(-4) \longrightarrow I_4(\Gamma) \longrightarrow 0.$$

So the Betti numbers are $\beta_{0,4}(I) = 5, \beta_{1,5}(I) = 3, \beta_{1,7}(I) = 4, \beta_{2,8}(I) = 4, \beta_{3,9}(I) = 1$ and $\beta_{i,j}(I) = 0$ otherwise, where $I = I_4(\Gamma)$.

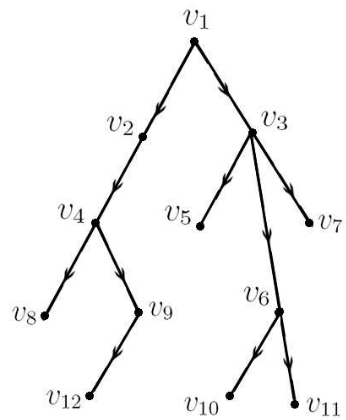

Figure 14

Definition 2.55. (***linear resolution***) We say that I has a linear resolution if there exists an integer $m \geq 1$ such that $\beta_{i,i+j}(I) = 0$ for all i and j with $j \neq m$.

Example 2.56. Let G be a cycle C_n. When $n = 4$, the edge ideal is $I(C_4) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1)$. The resolution of $I(C_4)$,

$$0 \longrightarrow R(-4) \longrightarrow R^4(-3) \longrightarrow R^4(-2) \longrightarrow I(C_4) \longrightarrow 0,$$

is a linear resolution. When $n = 6$, the edge ideal is $I(C_6) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1)$. The resolution of $I(C_6)$,

$$0 \longrightarrow R^2(-6) \longrightarrow R^6(-5) \longrightarrow R^6(-3) \oplus R^3(-4) \longrightarrow R^6(-2),$$

is not a linear resolution. (Please check the details about when the edge ideal $I(C_n)$ has a linear resolution for some n using the theorem of Fröberg in [**13**]).

Thus, in particular, if I has a linear resolution, then I is generated by homogeneous polynomials of the same degree. But the converse is not true. In example 2.54 the path ideal is generated by square-free monomials of the same degree, but the resolution is not linear.

CHAPTER 3

Cohen-Macaulay and Sequentially Cohen-Macaulay Rings and Modules

In the next chapter, we will prove that the ideal generated by all paths of length t in the tree Γ is ***sequentially Cohen-Macaulay***. The following is to introduce ***sequentially Cohen-Macaulay modules*** and their properties. Before that, we present some required ingredients.

1. Shellable simplicial complexes

We begin by introducing a class of simplicial complexes which are called shellable. Recall that a simplicial complex Δ of dimension $(d-1)$ is pure if all the facets of Δ have dimension $(d-1)$ i.e., $|F| = d$ for all facets.

Definition 3.1. A pure simplicial complex Δ is ***shellable*** if the facets of Δ can be listed $F_1, F_2, \ldots, F_n$ such that for all $1 \leq j < i \leq n$ there exists some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

Example 3.2. The simplicial complex $\Delta = \langle F_1, F_2, F_3 \rangle$, where $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_2, x_3, x_4\}$ and $F_3 = \{x_3, x_4, x_5\}$

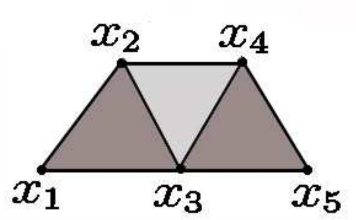

Figure 1

is shellable since

$$x_4 \in F_2 \setminus F_1 \text{ and } \{x_4\} = F_2 \setminus F_1$$

$$x_5 \in F_3 \setminus F_1 \text{ and } \{x_5\} = F_3 \setminus F_2$$

$$x_5 \in F_3 \setminus F_2 \text{ and } \{x_5\} = F_3 \setminus F_2.$$

The simplicial complex $\Delta = \langle F_1, F_2 \rangle$, where $F_1 = \{x_1, x_2, x_4\}, F_2 = \{x_3, x_4, x_5\}$

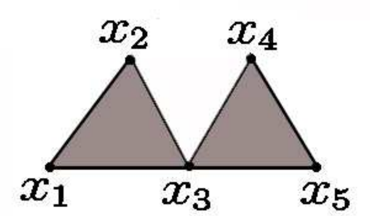

FIGURE 2

is not shellable since

$$x_1 \in F \setminus G, \text{ but } \{x_1\} \neq F \setminus G \text{ (or } G \setminus F)$$

$$x_2 \in F \setminus G, \text{ but } \{x_2\} \neq F \setminus G \text{ (or } G \setminus F)$$

$$x_5 \in G \setminus F, \text{ but } \{x_5\} \neq G \setminus F \text{ (or } F \setminus G).$$

An equivalent definition for a shellable complex is given below.

THEOREM 3.3. *A pure simplicial complex Δ is shellable if and only if the facets of Δ can be given a linear order $F_1, \cdots, F_n$ such that $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a nonempty set of maximal proper faces F_i for $i = 1, \ldots, n$.*

EXAMPLE 3.4. Consider the simplicial complex in Figure 1. Then we have
$\langle F_2 \rangle \cap \langle F_1 \rangle = \langle \{x_2, x_3\} \rangle \longleftarrow$ a maximal proper face of F_2
$\langle F_3 \rangle \cap \langle F_1, F_2 \rangle = \langle \{x_3\}, \{x_3, x_4\} \rangle = \langle \{x_3, x_4\} \rangle \longleftarrow$ a maximal proper face of F_3
So Δ is a shellable simplicial complex. Consider the simplicial complex of Figure 2. For this example, we have $\langle F \rangle \cap \langle G \rangle = \langle \{x_3\} \rangle$ which is not a maximal proper face of F or G. So this complex is not shellable. Note that the maximal proper faces of F are $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_1\}$.

Recall that if Δ is a simplicial complex, then the Stanley-Reisner ideal is

$$I_\Delta = (\{x_{i_1} \cdots x_{i_r} \ \mid\ \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta\}).$$

The quotient ring R/I_Δ is the Stanley-Reisner ring. The Stanley-Reisner ring of a shellable simplicial complex is a special type of ring; it is an example of a Cohen-Macaulay ring which is defined below.

2. Cohen-Macaulay rings

To define a Cohen-Macaulay (CM) ring, we need the notions of (Krull) dimension, regular sequences and depth.

DEFINITION 3.5. (***prime ideal, length of a chain***) A ***prime ideal*** of a ring S is an ideal $P \subsetneq S$ such that whenever $ab \in P$ then either $a \in P$ or $b \in P$. A chain of prime ideals is a strictly increasing sequence of prime ideals, i.e.

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subseteq S.$$

We say n is the ***length of the chain***.

DEFINITION 3.6. The ***(Krull) dimension of a ring***, denoted $\dim R$, is the length of the longest chain of prime ideals in R, i.e.

$$\dim R = \sup\{d \ \mid\ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d \subsetneq R, \text{ with } P_i \text{ prime}\}.$$

EXAMPLE 3.7. In $R = k[x_1, \ldots, x_n]$ any ideal generated by a subset of $\{x_1, \ldots, x_n\}$, e.g. $I = (x_{i_1}, \ldots, x_{i_r})$, is a prime ideal. So $\dim k[x_1, \ldots, x_n] \geq n$, since

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, x_2, \ldots, x_n)$$

is a chain of length n of prime ideals.

REMARK 3.8. $\dim k[x_1, \ldots, x_n] = n$, but this is nontrivial to show. See [**6**, Theorem 13.1].

THEOREM 3.9. [**18**, Theorem 22] *If Δ is a simplicial complex, then*

$$\dim R/I_\Delta = \dim \Delta + 1$$

where by $\dim R/I_\Delta$ *we mean the dimension of the ring, and by* $\dim \Delta$ *we mean the dimension of the simplicial complex.*

This theorem is useful to compute the dimension of the quotient ring R/I_{Δ}, when the square-free monomial I_{Δ} is a Stanley-Reisner ideal associated to a simplicial complex.

EXAMPLE 3.10. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) = (x_1, x_2) \cap (x_3, x_4)$ in $R = k[x_1, x_2, x_3, x_4]$. We will compute the dimension of R/I.

First, recall that $\mathcal{P}$ is a prime ideal in R/I if and only if there exists a prime ideal $I \subseteq P \subsetneq R$ such that $\mathcal{P} = P/I$. Also, note that if P is any prime ideal with $I \subseteq P$, then either

(1) $x_1, x_2 \in P$ or

(2) $x_3, x_4 \in P$.

Set $\mathcal{P}_0 = (x_1, x_2)/I$, $\mathcal{P}_1 = (x_1, x_2, x_3)/I$, and $\mathcal{P}_2 = (x_1, x_2, x_3, x_4)/I$. Then $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2$ is a chain of prime ideals in R/I, so it follows that $\dim R/I \geq 2$.

Suppose there is a chain $\mathcal{Q}_0 \subsetneq \mathcal{Q}_1 \subsetneq \cdots \subsetneq \mathcal{Q}_n \subsetneq R/I$ with $n \geq 3$. So $\mathcal{Q}_i = Q_i/I$ for some prime ideal $I \subsetneq Q_i \subsetneq R$. Thus, we have a chain

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq R.$$

Suppose we are in case (1), i.e., $x_1, x_2 \in Q_0$. Then

$$(0) \subsetneq (x_1) \subsetneq Q_0 \subsetneq \cdots \subsetneq Q_n.$$

is a chain of length $n+2 \geq 3+2 = 5$ in R. This contradicts the fact that $\dim R = 4$. A similar argument for case (2) will give us a similar conclusion. Thus $\dim R/I \leq 2$. Hence, $\dim R/I = 2$.

DEFINITION 3.11. ***(zero divisor)*** [**4**] A ***zero divisor*** of a ring R is an element $a \in R$ such that $a \neq 0$ and there exists $0 \neq b \in R$ such that $ab = 0$.

DEFINITION 3.12. (***regular***) Let $I \subset R = k[x_1, \cdots, x_n]$. An element $F \in R$ is a ***regular*** element on R/I if $\bar{F} = (F + I)$ is not a zero divisor of R/I. Equivalently, F is ***regular*** on R/I if whenever $FG \in I$, then $G \in I$.

EXAMPLE 3.13. Consider any $x_i \in R = k[x_1, \ldots, x_n]$. Then x_i is regular on $R = R/(0)$ since R is a domain.

EXAMPLE 3.14. Suppose $I = (xyz) \subseteq R = k[x, y, z]$. Then xy is not regular on R/I since $\overline{xy} \neq \overline{0} \in R/I$ and $\overline{z} \neq \overline{0} \in R/I$ but $\overline{xy}(\overline{z}) = \overline{xyz} = \overline{0}$ in R/I.

Example 3.15. Let $I = (x_1, x_2) \cap (x_3, x_4) \subset k[x_1, x_2, x_3, x_4] = R$. We show that $(x_1 + x_3)$ is regular on R/I.

Suppose $(x_1 + x_3)G \in J = (x_1, x_2) \cap (x_3, x_4)$. So $(x_1 + x_3)G \in (x_1, x_2)$ and $(x_1 + x_3)G \in (x_3, x_4)$. Both (x_1, x_2) and (x_3, x_4) are prime ideals. Also $(x_1 + x_3) \notin (x_1, x_2)$ and (x_3, x_4). So $G \in (x_1, x_2) \cap (x_3, x_4) = I$.

Definition 3.16. (***regular sequence***) A sequence $F_1, \ldots, F_m$ of R is called a ***regular sequence*** on R/I if

(1) $\overline{F_1}$ is regular on R/I, and
(2) $\overline{F_i}$ is regular on $R/(I, F_1, \cdots, F_{i-1})$, for $i = 2, \ldots, m$.

Example 3.17. If $R = k[x_1, \ldots, x_n]$ and $I = (0)$, then $x_1, \ldots, x_n$ is a regular sequence on R/I since

(1) $\overline{x_1}$ is regular on $R/(0)$.
(2) $\overline{x_i}$ is regular on $R/(x_1, \ldots, x_{i-1}) \cong k[x_i, \ldots, x_n]$.

Theorem 3.18. [**6**, Corollary 17.2] *All maximal regular sequence have the same length, and any regular sequence can be extended to a maximal regular sequence.*

Definition 3.19. (***depth***) The ***depth*** of R/I, denoted $depth(R/I)$, is the length of the longest maximal regular sequence on R contained in $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$.

Theorem 3.20. *(Auslander-Buchsbaum Formula)* [**6**, Theorem 19.9]. *Let $R = k[x_1, \ldots, x_n]$, and $I \subseteq R$ a homogeneous ideal. Then*

$$pd(R/I) + depth(R/I) = n.$$

Theorem 3.21. [**6**, Theorem 19.1] *For any ideal $I \subseteq k[x_1, \ldots, x_n] = R$,* $\operatorname{depth}(R/I) \leq \dim(R/I)$.

Definition 3.22. (***Cohen-Macaulay ring***) A ring R/I is ***Cohen-Macaulay*** if $\operatorname{depth}(R/I) = \dim(R/I)$.

Example 3.23. The ring $R = k[x_1, \ldots, x_n]$ is Cohen-Macaulay since $\operatorname{depth}(R/I) = \dim(R/I) = n$.

Example 3.24. If $I = (x_1, x_2) \cap (x_3, x_4) \subseteq R = k[x_1, \ldots, x_4]$, then R/I is not Cohen-Macaulay. We saw that $\dim R/I = 2$ and $x_1 + x_3$ is regular on R/I. So $1 \leq \operatorname{depth}(R/I)$. We want to show that $\operatorname{depth}(R/I) = 1$.

Take any $G \in \mathfrak{m} = (x_1, x_2, x_3, x_4)$. We need to show G cannot be regular on $R/(I, x_1 + x_3)$. We can write G as

$$G = G_1(x_1, x_2, x_3, x_4)x_1 + G_2(x_2, x_3, x_4)x_2 + G_3(x_3, x_4)x_3 + G_4(x_4)x_4.$$

Suppose $\overline{G} \neq \overline{0}$ in $R/(I, x_1 + x_3)$. This implies that $G \notin (I, x_1 + x_3)$. Note that $\overline{x_1} \neq \overline{0} \in R/(I, x_1 + x_3)$. But $Gx_1 = G_1 x_1^2 + G_2 x_1 x_2 + G_3 x_3 x_1 + G_4 x_4 x_1$.

Now

$$\begin{aligned}
x_1^2 = x_1(x_1 + x_3) - x_1x_3 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3) \\
x_1x_2 = x_2(x_1 + x_3) - x_2x_3 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3) \\
x_3x_1 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_2 + x_3) \\
x_4x_1 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3).
\end{aligned}$$

So $Gx_1 \in (I, x_1 + x_3)$ but $G \notin (I, x_1 + x_3)$. Thus G is not regular. Therefore we cannot extend the length of the regular sequence. So $\operatorname{depth}(R/I) = 1$.

We now relate Cohen-Macaulay with the notion of shellable introduced at the beginning of this chapter.

Theorem 3.25. [**19**, Theorem 13.45] *Suppose that Δ is a shellable simplicial complex. If R/I_Δ is the associated Stanley-Reisner ring, then R/I_Δ is Cohen-Macaulay.*

Example 3.26. Let Δ be the simplicial complex in Figure 3.

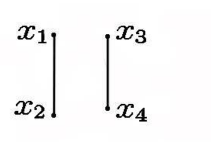

Figure 3

Then $I_\Delta = (x_1x_3, x_1x_4, x_2x_3, x_3x_4) = (x_1, x_2) \cap (x_3, x_4)$. This simplicial complex Δ is not shellable since R/I_Δ is not Cohen-Macaulay as shown in Example 3.24.

Definition 3.27. (***Cohen-Macaulay simplicial complex***) We call Δ a ***Cohen-Macaulay simplicial complex*** if R/I_Δ is Cohen Macaulay.

The following theorem describes the relationship between a Cohen-Macaulay ring and its Alexander dual when I is square-free.

THEOREM 3.28. *R/I_Δ is Cohen-Macaulay if and only if $I_\Delta^\vee$ has a linear resolution.*

Recall the definition of the linear resolution of Definition 2.55. Let I be a homogeneous ideal of R, and suppose that all the minimal generators of I have the same degree, say d. We say that I has a linear resolution if for all $i \geq 1$, $\beta_{i,j}(R/I) = 0$ if $j \neq i + d - 1$. Let us see an example of Theorem 3.28.

EXAMPLE 3.29. Let G be a pentagon. The edge ideal is

$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1).$$

Using CoCoA to compute the depth of $R/I(G)$ and the Krull dimension of $R/I(G)$, we get $depth(R/I(G)) = \dim(R/I(G)) = 2$. Hence G is Cohen-Macaulay. The Alexander dual of $I(G)$ is

$$\begin{aligned} I(G)^\vee &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \\ &= (x_2x_3x_5, x_1x_3x_5, x_1x_2x_4, x_1x_3x_4, x_2x_4x_5). \end{aligned}$$

And the resolution of $I(G)^\vee$ is

$$0 \longrightarrow R(-5) \longrightarrow R^5(-4) \longrightarrow R^5(-3) \longrightarrow R \longrightarrow I(G)^\vee \longrightarrow 0.$$

This shows the Alexander dual of $I(G)$ has a linear resolution.

3. Sequentially Cohen-Macaulay modules and componentwise linearity

DEFINITION 3.30. (***Sequentially Cohen-Macaulay***) Let M be a graded module over $R = k[x_1, \dots, x_n]$. We call M ***sequentially Cohen-Macaulay (SCM)*** if there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

of M by graded *R-modules* such that M_i/M_{i-1} is Cohen-Macaulay for all i, and $\dim M_i/M_{i-1} < \dim M_{i+1}/M_i$ for all i, where dim denotes the Krull dimension.

DEFINITION 3.31. If I is an homogenous ideal, let (I_d) denote the ideal generated by all the degree d elements of I.

EXAMPLE 3.32. $I = (x_1, x_2^3) \subseteq k[x_1, x_2]$. Then

$$\begin{aligned}(I_1) &= (x_1)\\ (I_2) &= (x_2^2, x_1x_2)\\ (I_3) &= (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3) \subseteq (x_1, x_2)\\ (I_4) &= (x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4) \subseteq (x_1, x_2)\end{aligned}$$

DEFINITION 3.33. (***componentwise linear***)[**17**] An ideal I is *componentwise linear* if (I_d) has a linear resolution for all positive integer $d \in \mathbb{N}$.

THEOREM 3.34. [**17**, Theorem 2.1] *Let I be a square-free monomial ideal of R. Then R/I is sequentially Cohen-Macaulay if and only if $I^\vee$ is componentwise linear.*

EXAMPLE 3.35. Use the ideal in Example 3.32. We find the resolution for all (I_d), where $I \subseteq R = k[x_1, x_2]$.

- $\mathrm{Res}((I_1)): \ 0 \to R(-1) \to (I_1) \to 0$
- $\mathrm{Res}((I_2)): \ 0 \to R(-3) \to R^2(-2) \to (I_2) \to 0$
- $\mathrm{Res}((I_3)): \ 0 \to R^3(-4) \to R^4(-3) \to (I_3) \to 0$

When $d \geq 3$, we have $(I_d) = (x_1, x_2)_d$. This is because when $d \geq 3$ every generator of (I_d) has the form $x_1^i x_2^{d-i}$, which is in the ideal $(x_1, x_2)_d$, i.e. $(I_d) \subseteq (x_1, x_2)_d$. The converse is that $(x_1, x_2)_d = (\{x_1^i x_2^{d-i} \mid i = 0, 1, \ldots d\})$. So when $d \geq 3$, we have $(x_1, x_2)_d \subseteq (I_d)$. Then since we know $k[x_1, x_2]/(x_1, x_2)$ is Cohen-Macaulay, it is also sequentially Cohen-Macaulay (SCM). Hence by Theorem 3.34 we have that $(x_1, x_2)_d = (I_d)$ has a linear resolution for all $d \geq 3$.

The theorem below will give an important result about simplicial tree which will be used in this paper.

THEOREM 3.36. ***(Simplicial trees are SCM)*** [**11**, Corollary 5.6]. *The facet ideal of a simplicial tree is sequentially Cohen-Macaulay.*

The primary idea in Faridi's proof is to prove that if $\mathcal{I}(\Delta)$ is the facet ideal of a simplicial tree Δ, then the Alexander dual $\mathcal{I}(\Delta)^\vee$ is componentwise linear. Then, by Theorem 3.34, $R/I(\Delta)$ is SCM. This approach was also used by Francisco and Van Tuyl to prove that $R/I(G)$ is SCM when G is chordal.

CHAPTER 4

The Simplicial Complex associated to $I_t(\Gamma)$ is a simplicial tree

In this chapter, we will study the properties of the path ideal of a tree. In particular, we will show that $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay.

1. The path ideal of a tree is sequentially Cohen-Macaulay

Recall that in a rooted tree the ***length*** of a path is the number of edges in the path, and is denoted by length(F), where F is a path of a rooted tree.

Definition 4.1. (***height***) The ***height*** of a rooted tree Γ, denoted $height(\Gamma)$, is the length of the longest path starting at the root of the tree.

Example 4.2. For the tree in Figure 1, $v_1v_2v_4v_8$ is a path of length 3, and the edge v_3v_7 is a path of length 1. The height of this tree is 3 since the length of the longest path is 3.

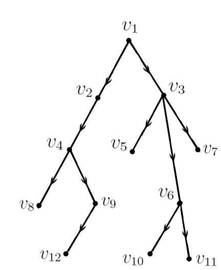

Figure 1

Definition 4.3. (***path ideals of a directed tree***) The ideal generated by all paths of length $t-1$ in a rooted directed tree Γ, denoted $I_t(\Gamma)$, is the monomial ideal:

$$I_t(\Gamma) = (\{x_{i_1} \cdots x_{i_t} \ \mid \ x_{i_1} \cdots x_{i_t} \text{ is a path of length } t-1 \text{ in } \Gamma\}).$$

It follows that if Γ is a tree with $height(\Gamma) < t-1$, then $I_t(\Gamma) = (0)$.

We can associate to Γ a simplicial complex. Define the simplicial complex

$$\Delta_t(\Gamma) = \langle \{x_{i_1}, \ldots, x_{i_t}\} \ \mid \ x_{i_1} \cdots x_{i_t} \text{ is a path of length } t-1 \text{ in } \Gamma \rangle.$$

The simplicial complex $\Delta_t(\Gamma)$ and ideal $I_t(\Gamma)$ are related as follows.

LEMMA 4.4. *Let Γ be a rooted tree, $t \geq 2$ and consider the simplicial complex*

$$\Delta_t(\Gamma) = \langle \{x_{i_1}, \ldots, x_{i_t}\} \ \mid \ x_{i_1} \cdots x_{i_t} \textit{ is a path of length } t-1 \textit{ in } \Gamma \rangle.$$

Then $\mathcal{I}(\Delta_t(\Gamma)) = I_t(\Gamma)$, i.e., $I_t(\Gamma)$ is the facet ideal of $\Delta_t(\Gamma)$.

PROOF. The simplicial complex $\Delta_t(\Gamma)$ is pure (Definition 1.19) and every facet has dimension $t-1$. Take a facet $\{x_{i_1}, \ldots, x_{i_t}\} \in \Delta_t(\Gamma)$. By the definition of $\Delta_t(\Gamma)$, we know $x_{i_1} \cdots x_{i_t}$ is a path of length $t-1$. So $x_{i_1} \cdots x_{i_t} \in I_t(\Gamma)$. This implies $\mathcal{I}(\Delta_t(\Gamma)) \subseteq I_t(\Gamma)$. Take a generator $x_{i_1} \cdots x_{i_t}$ of $I_t(\Gamma)$ and we have $\{x_{i_1}, \ldots, x_{i_t}\}$ is a facet in $\Delta_t(\Gamma)$. So $\{x_{i_1}, \ldots, x_{i_t}\} \in \mathcal{I}(\Delta_t(\Gamma))$. Then $\mathcal{I}(\Delta_t(\Gamma)) \supseteq I_t(\Gamma)$. □

In the following lemma we prove the intersection of any two paths must start at the first vertex of one of these two paths.

LEMMA 4.5. *The intersection of any two distinct paths F and G in a directed rooted tree Γ is a path of length $|F \cap G| - 1$ starting at the first vertex of either F or G.*

PROOF. First we want to show that the intersection of any two paths must start at the first vertex of one of these two paths. Assume $F \cap G$ starts at v_m which is neither the first vertex of F nor G. So, there is a vertex $u \in F$ and $w \in G$ such that uv_m is a directed edge in the path F and wv_m is a directed edge in G. We know all the paths start at the root v_0. Then there are two paths between v_0 and v_m through F and G respectively. This contradicts the definition of a tree. So $F \cap G$ start at the first vertex of F or G.

Second, we will show $F \cap G$ is a connected path of length $|F \cap G| - 1$. Let F be a path in a tree Γ, and suppose $F = v_{i_1} \cdots v_{i_t}$. Suppose $F \cap G = v_{i_{m_1}} \cdots v_{i_{m_s}}$ for some path G. We claim that $F \cap G$ is a connected path in Γ. Suppose $F \cap G = v_{i_{m_1}} \cdots v_{i_{m_j}} \cup v_{i_{m_k}} \cdots v_{i_{m_s}}$ are two disjoint paths in Γ. Thus there is a path $G = v_{i_{m_1}} \cdots v_{i_{m_j}} v_{g_{j+1}} \cdots v_{g_{k-1}} v_{i_{m_k}} \cdots v_{i_{m_s}}$ in Γ, where $G \neq F$. But then there are two different paths $v_{m_{j+1}} \cdots v_{m_{k-1}} \subset F$ and $v_{g_{j+1}} \cdots v_{g_{k-1}} \subset G$ between $v_{i_{m_j}}$ and $v_{i_{m_k}}$. This contradicts the definition of a tree because for any two vertices there is only one path connecting the two vertices. Thus the intersection of F and G is a path with $|F \cap G| - 1$ edges, i.e. $length(F \cap G) = |F \cap G| - 1$. □

DEFINITION 4.6. (***level***) The ***level*** of a vertex v in a rooted tree Γ, denoted $level(v)$, is the length of the path starting at the root and ending at v.

From Lemma 4.5 above we get the following corollary.

COROLLARY 4.7. *The intersection of any two paths is a path starting at the lower starting vertex of the two paths. Equivalently we say it starts at the starting vertex with larger level.*

LEMMA 4.8. *Let F be a path of length $t-1$ in Γ that contains a leaf of Γ. Then F corresponds to a leaf in the simplicial complex $\Delta_t(\Gamma)$.*

PROOF. Let $F = v_{i_1} \cdots v_{i_t}$ be a path of length $t-1$ in Γ and suppose v_{i_t} is a leaf. For any facet $G_j \in \Delta_t(\Gamma)$, G_j is a path of length $t-1$. Since v_{i_t} is free, we have $G_j \cap F \subseteq v_{i_1} \cdots v_{i_{t-1}}$, for any j. If there is a vertex v_{i_0} with smaller level than that of v_{i_1} and connects with v_{i_1}, we choose the facet $G = v_{i_0} \cdots v_{i_{t-1}}$ and we have $G_j \cap F \subseteq G \cap F = v_{i_1} \cdots v_{i_{t-1}}$. This implies F is a leaf in $\Delta_t(\Gamma)$.

If there is no vertex with smaller level and connects with v_{i_1}, then the vertex v_{i_1} is the root of the tree Γ. Set $\bigcup_{G_j \neq F, G_j path}(G_j \cap F) = v_{i_1} \cdots v_{i_m}$, $m < t$. There must be a path G_j of length $t-1$ through v_{i_m}. The path looks like $G_j = v_{i_j} \cdots v_{i_m} v_{r_{m+1}} \cdots v_{r_k}$, $1 \leq j \leq m < t \leq k$ and $k = t + j - 1$ since G_j is of length $t-1$. Since $G_j \cap F \subseteq v_{i_1} \cdots v_{i_m}$, $length(G_j \cap F) \leq m - 1$. So the length of the remaining part $v_{r_{m+1}} \cdots v_{r_k}$ of the path G_j, which is not in the union of the intersections of F with any other facets, is bigger than $t - m$. Now we get a path $N = v_{i_1} \cdots v_{i_m} v_{r_{m+1}} \cdots v_{r_k}$ of length greater than t. We pick a subsequence of N to get a path $G = v_{i_1} \cdots v_{i_m} v_{r_{m+1}} \cdots v_{r_t}$ in Γ such that $G_j \cap F \subseteq G \cap F = v_{i_1} \cdots v_{i_m}$. Thus F is a leaf of $\Delta_t(\Gamma)$. $\square$

THEOREM 4.9. *Let Γ be a rooted tree and $t \geq 2$ a positive integer. Then $\Delta_t(\Gamma)$ is a simplicial tree.*

PROOF. There are 3 cases to consider:

Case 1: If $height(\Gamma) < t - 1$, then $I_t(\Gamma) = (0)$. This is the trivial case.

Case 2: If $height(\Gamma) = t - 1$, then all the generators of $I_t(\Gamma)$ have a leaf in Γ and start at the root of Γ. Then by Lemma 4.8, all the paths correspond to leaves of the simplicial complex Δ. Thus Δ is a simplicial tree in this case because all subcomplexes will have a leaf.

Case 3: If $height(\Gamma) > t - 1$, we want to show that the simplicial complex $\Delta_t(\Gamma)$ is a simplicial tree, i.e. every subcomplex of $\Delta_t(\Gamma)$ contains a leaf. By Definition 1.30 we need to show that given any subset $\langle G_1, \ldots, G_l \rangle$ of the facet set of $\Delta_t(\Gamma)$, the corresponding subcomplex has a leaf (this is the definition of a simplicial tree). We do induction on $n = height(\Gamma)$.

The base case is true since if $heigth(\Gamma) = n \leq t - 1$, we have already seen that $\Delta_t(\Gamma)$ is a simplicial tree. Assume the statement is true when $n = k$. That is to say that for all trees Γ with $height(\Gamma) \leq n$, the simplicial complex $\Delta_t(\Gamma)$ is a simplicial tree, i.e., every subcomplex of $\Delta_t(\Gamma)$ has a leaf. So assume Γ is a tree of height $n + 1$.

If $\langle G_1, \ldots, G_l \rangle \subseteq \Delta_t(\Gamma)$ is a subset that contains a facet G_i that contains a leaf of Γ, then G_i is a leaf by Lemma 4.8.

On the other hand, suppose every G_i does not contain a leaf of Γ. Then G_i is still an element of $\Delta_t(\Gamma')$ where $\Gamma' = \Gamma \setminus \{v_{i_1}, \ldots, v_{i_m}\}$. The vertices $v_{i_1}, \ldots, v_{i_m}$ are all the leaves of Γ. Now Γ' is a tree of height n. By the induction hypothesis, we have $\Delta_t(\Gamma')$ is a simplicial tree, i.e. there exists a facet in $\langle G_1, \ldots, G_l \rangle$ that must be a leaf since $height(\Gamma') = n$. This shows there exists an facet in $\langle G_1, \ldots, G_l \rangle$ that is a leaf. So the simplicial complex $\Delta_t(\Gamma)$ is a simplicial tree since every subcomplex $\langle G_1, \ldots, G_l \rangle$ has a leaf. $\square$

The above theorem shows that when we view the path ideal $I_t(\Gamma)$ of the tree Γ as the facet ideal of a simplicial complex, this simplicial complex is a simplicial tree. We will apply a result of Faridi's paper [**11**, Corollary 5.6] to prove:

COROLLARY 4.10. *If $I_t(\Gamma)$ is a path ideal of a tree Γ, then $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay for all $t \geq 2$.*

PROOF. By Lemma 4.4, we know that the facet ideal of the simplicial complex

$$\Delta_t(\Gamma) = \langle \{x_{i_1}, \ldots, x_{i_t}\} \ \mid \ x_{i_1} \cdots x_{i_t} \text{ is a path of length } t-1 \text{ in } \Gamma \rangle$$

is $I_t(\Gamma)$. Theorem 4.9 shows this simplicial complex $\Delta_t(\Gamma)$ is a simplicial tree. Applying the property that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay (see Theorem 3.36), we get $R/I_t(\Gamma)$ is sequentially Cohen-Macaulay. $\square$

2. Properties of a path ideal

In this section, we investigate the graph of a line. Further properties on this line can be found in the Ph.D. thesis of Jacques [**16**].

DEFINITION 4.11. A ***line graph*** L_n is the graph on the vertex set $V = \{x_1, \ldots, x_n\}$ with edge set $E_{L_n} = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$. When the vertex x_1 is assigned the root, and all the edges have the direction away from the root, the line L_n is a ***directed line***.

EXAMPLE 4.12. The line graph looks like:

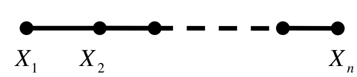

FIGURE 2

THEOREM 4.13. *Consider the line graph L_n. Then the path ideal $I_n(L_n) = (x_1, \ldots, x_n)$ of the line L_n is CM, i.e., $R/I_n(L_n)$ is Cohen-Macaulay.*

Proof. Since the resolution of $R/I_n(L_n)$ is

$$0 \longrightarrow R(-n) \longrightarrow R \longrightarrow R/I_n(L_n) \longrightarrow 0,$$

we know the projective dimension of $R/I_n(L_n)$ is always 1. Then the depth of $R/I_n(L_n)$ is $n-1$ by the Auslander-Buchsbaum formula 3.20 (Theorem 19.9 [**22**]).

On the other hand, if we let $I_n(L_n)$ be a Stanley-Reisner ideal of a simplicial complex $\Delta = \langle F_1, \ldots, F_t \rangle$, each facet F_i has dimension $n-2$. This is because Δ is on V and $x_1 \cdots x_n$ is the largest and the only nonface in Δ. All the other faces of smaller dimension are in Δ. So all the facets contain $n-1$ vertices. That is, $\dim F_i = |F_i| - 1 = n - 1 - 1 = n - 2$. Thus $\dim \Delta = n - 2$. By Theorem 3.9, $\dim(R/I_n(L_n)) = n - 1$. We thus get $\dim(R/I_n(L_n)) = \operatorname{depth}(R/I_n(L_n))$. So $R/I_n(L_n)$ is Cohen-Macaulay. $\square$

Below are some values of the projective dimensions (see the definition of projective dimension in Definition 2.52) of path ideals for lines. For each L_n, the projective dimension was computed using CoCoA. A table of our results is given below.

n	2	3	4	5	6	7	8	9	10
$\mathrm{pd}(I_2(L_n))$	1	2	2	3	4	4	5	6	6
$\mathrm{pd}(I_3(L_n))$	0	1	2	2	2	3	4	4	4
$\mathrm{pd}(I_4(L_n))$	0	0	1	2	2	2	2	3	4
$\mathrm{pd}(I_5(L_n))$	0	0	0	1	2	2	2	2	2
$\mathrm{pd}(I_6(L_n))$	0	0	0	0	1	2	2	2	2
$\mathrm{pd}(I_7(L_n))$	0	0	0	0	0	1	2	2	2
$\mathrm{pd}(I_8(L_n))$	0	0	0	0	0	0	1	2	2
$\mathrm{pd}(I_9(L_n))$	0	0	0	0	0	0	0	1	2
$\mathrm{pd}(I_{10}(L_n))$	0	0	0	0	0	0	0	0	1

In this table every column shows all the projective dimensions of the path ideals $I_t(L_n)$ of a line L_n for all $t \geq 2$. When $t > n$, there is no generator of $I_t(L_n)$ because t is greater than the number of vertices of the vertex set V. So, we get $I_t(L_n) = (0)$. Therefore $pd(I_t(L_n)) = 0$, for $t > n$. All the entries are 0's in the bottom left corner.

Jacques gave a formula for the first row of this table, i.e., for the edge ideal $I_2(L_n)$. He gives the formulas of projective dimensions for all $n \geq 2$.

THEOREM 4.14. [**16**] *The projective dimension of the line graph is independent of the characteristic of the chosen field and is*

$$pd(I(L_n)) = \begin{cases} \frac{2n}{3}, & if n \equiv 0 \mod 3 \\ \frac{2n-2}{3}, & if n \equiv 1 \mod 3 \\ \frac{2n-1}{3}, & if n \equiv 2 \mod 3. \end{cases}$$

In Theorem 4.13 it was shown that when $t = n$ the path ideal $I_t(L_n)$ is Cohen-Macaulay. Also, it was shown that $pd(I_t(L_n)) = 1$. So all entries in the diagonal of this table are equal to 1. The following theorem will show that in the 2^{nd} diagonal all the elements are 2's.

THEOREM 4.15. *For a line L_n on $V = \{x_1, \dots, x_n\}$, the projective dimension of the path ideal $I_{n-1}(L_n)$ is 2.*

PROOF. The path ideal $I_{n-1}(L_n)$ has form $I_{n-1}(L_n) = (x_1F, x_nF)$, where $F = x_2x_3 \cdots x_{n-1}$. Consider the map $\phi_0 : R^2 \longrightarrow R$ given by

$$\phi_0 \left(\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \right) = G_1x_1F + G_2x_nF.$$

The kernel of ϕ_0 is

$$ker\phi_0 = \left\{ \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} | G_1x_1F + G_2x_nF = 0 \right\} = \left\{ \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} | F(G_1x_1 + G_2x_n) = 0 \right\}.$$

Since $F = x_2x_3 \cdots x_{n-1}$ is not a zero divisor, we have $G_1x_1 + G_2x_n = 0$. From this it follows $G_1x_1 = -G_2x_n$, i.e. $x_1|G_2$ and $x_n|G_1$. So

$$ker\phi_0 = \left\{ F \begin{bmatrix} -x_n \\ x_1 \end{bmatrix} | F \in R \right\}.$$

The resolution of $I_{n-1}(L_n)$ is

$$0 \longrightarrow R \xrightarrow{\phi_1} R^2 \xrightarrow{\phi_0} I_{n-1}(L_n) \longrightarrow 0$$

where

$$\phi_1 : R \longrightarrow R^2$$

$$G \mapsto G \begin{bmatrix} -x_n \\ x_1 \end{bmatrix} = \begin{bmatrix} -Gx_n \\ Gx_1 \end{bmatrix}.$$

To see this, we need to prove that $Im\phi_1 = ker\phi_0$. For any $G \in R$, we have

$$\begin{bmatrix} -Gx_n \\ Gx_1 \end{bmatrix} \in Im\phi_1,$$

and

$$\phi_0(\phi_1(G)) = \phi_0\left(\begin{bmatrix} -Gx_n \\ Gx_1 \end{bmatrix}\right) = -Gx_1x_nF + Gx_1x_nF = 0.$$

So $Im\phi_1 \subseteq ker\phi_0$. If $\begin{bmatrix} A \\ B \end{bmatrix} \in ker\phi_0$, then

$$\phi_0\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = Ax_1F + Bx_nF = 0.$$

This follows $Ax_1F = -Bx_nF$. This shows $x_n|A$ and $x_n|B$. Let A be Mx_n and B be Nx_1 and plug in $Ax_1F + Bx_nF = Mx_1x_nF + Nx_1x_nF = 0$. We get $M = -N$. So

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -Nx_n \\ Nx_1 \end{bmatrix} \in Im\phi_1$$

I.e. we showed $Im\phi_1 \supseteq ker\phi_0$. Hence, $Im\phi_1 = ker\phi_0$. So the projective dimension of $I_{n-1}(L_n) = 2$. □

CHAPTER 5

Open questions

I gave the main results about the properties of the path ideal in a directed rooted tree in Chapter 4. There are many interesting questions that one can still ask about path ideals and their properties. The list below shows some of the open questions about the properties of the path ideal.

(1) Find formulas for the projective dimension of the path ideal of lines.
(2) Study the ring $R/I_t(G)$ when G is a cycle graph or bipartite graph.
(3) Give the minimal graded free resolution of path ideals.

For the first open question, Sean Jacques, in [Corollary 7.7.35, [**16**]], gave formulas for the projective dimensions of the edge ideals of the line graph L_n , i.e., $I_t(\Gamma)$ where $t = 2$. (See Theorem 4.14). I would like to find formulas for $pd(I_t(L_n))$ for $t > 2$.

Currently it's hard to find formulas for the projective dimension for every path ideal in a tree. More data is still required. For my first step, I will use CoCoA to compute more projective dimensions and give a bigger table. I then hope to prove the numbers in the 3^{rd} diagonal are all 2's.

Recently some mathematicians have been interested in classifying or identifying sequentially Cohen-Macaulay graphs in terms of the combinatorial properties of the graph. In [**23**] Van Tuyl and Villarreal investigate what families of graphs have the properties that the graph is shellable. They also classify all the shellable bipartite graphs.

Theorem 5.1. [**23**] *Let G be a bipartite graph. Then G is sequentially Cohen-Macaulay if and only if G is shellable.*

One can also ask the question "Are the simplicial complexes $\Delta_t(\Gamma)$ defined by a path ideal $I_t(\Gamma)$ of a tree Γ shellable?" Since Γ is a bipartite graph, answering this question would partially generalize the above Theorem 5.1.

In [**1**] Brumatti and da Silva determine for what lengths $t-1$ of the paths in a cycle $G = C_n$ the path ideal $I_t(G)$ is of linear type. I am interested in determining when $I_t(G)$ with $t \geq 3$ is SCM or CM when $G = C_n$, a cycle of length n. (Reference for $t = 2$ is [**7**, Proposition 4.1]). Francisco and Van Tuyl gave

$$I_2(C_n)\ is\ \begin{cases} CM, & \text{iff } n = 3, 5; \\ SCM, & \text{iff } n = 3, 5. \end{cases}$$

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