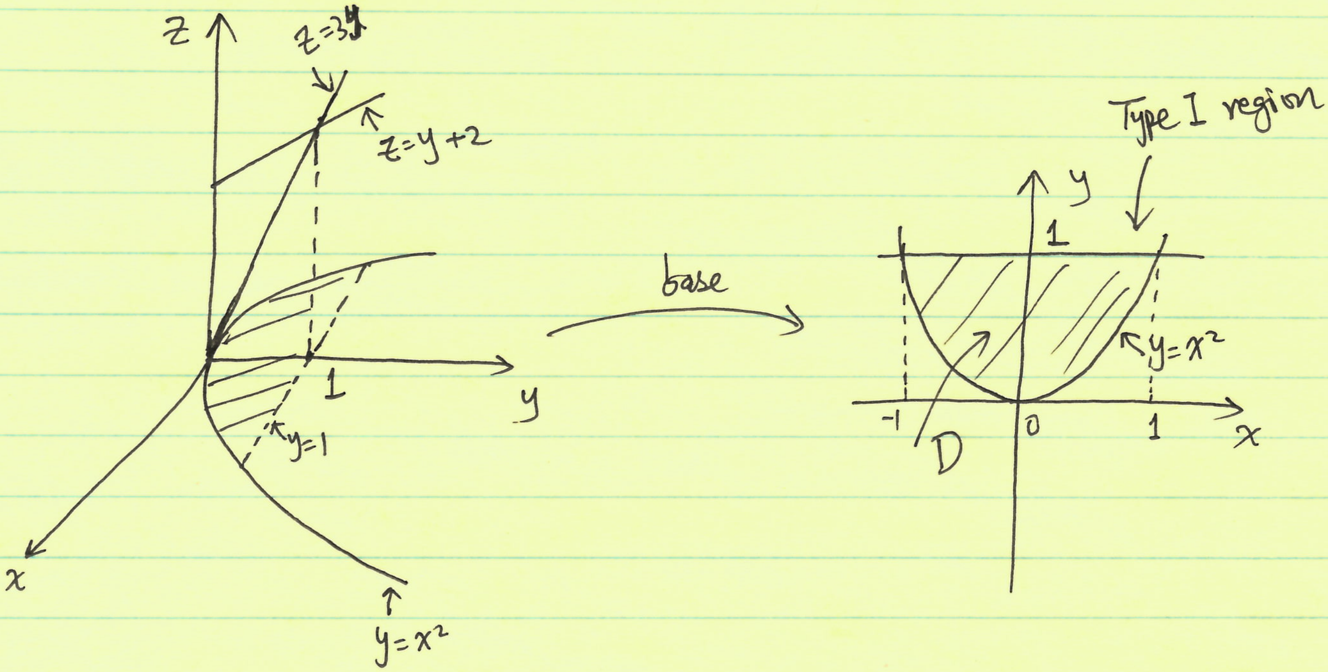
Homework 1. Solution:

1. (§15.2, #36).



From the above picture, we see that the solid S enclosed by the parabolic cylinder $y=x^2$ and the planes $z=3y$, $z=2+y$ can be described as the difference between two solids: $S = S_1 - S_2$,

where S_1 is the solid lies under the graph of $z=y+2$ and above $D=\{(x,y) \mid -1\le x\le 1,\ x^2\le y\le 1\}$,

S_2 is the solid lies under the graph of $z=3y$ and above $D=\{(x,y) \mid -1\le x\le 1,\ x^2\le y\le 1\}$.

i.e. $S=\{(x,y,z) \mid 3y\le z\le y+2,\ -1\le x\le 1,\ x^2\le y\le 1\}$

$S_1=\{(x,y,z) \mid 0\le z\le y+2,\ -1\le x\le 1,\ x^2\le y\le 1\}$

$S_2=\{(x,y,z) \mid 0\le z\le 3y,\ -1\le x\le 1,\ x^2\le y\le 1\}$

Thus, $V(S) = V(S_1) - V(S_2)$

$$= \iint_D (y+2)\,dA - \iint_D 3y\,dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (y+2)\,dy\,dx - \int_{-1}^{1}\int_{x^2}^{1} 3y\,dy\,dx$$

$$= \int_{-1}^{1} \left(\frac{y^2}{2}+2y\right)\Big|_{y=x^2}^{y=1} dx - \int_{1}^{-1} \frac{3}{2}y^2\Big|_{y=x^2}^{y=1} dx$$

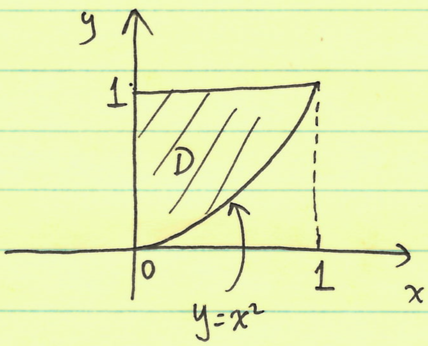
$$= \int_{-1}^{1} \left(\frac{5}{2} - \frac{x^4}{2} - 2x^2\right) dx - \int_{-1}^{1} \left(\frac{3}{2} - \frac{3}{2}x^4\right) dx$$

$$= \left(\frac{5}{2}x - \frac{x^5}{10} - \frac{2}{3}x^3\right)\Big|_{-1}^{1} - \left(\frac{3}{2}x - \frac{3}{10}x^5\right)\Big|_{-1}^{1}$$

$$= \left(5 - \frac{1}{5} - \frac{4}{3}\right) - \left(3 - \frac{3}{5}\right)$$

$$= 2 + \frac{2}{5} - \frac{4}{3} = \frac{30+6-20}{15} = \frac{16}{15}.$$

2. (§15.2, #52).



D is both $\underbrace{\text{Type I}}$ and $\underbrace{\text{Type II}}$ region.

$D=\{(x,y) \mid 0\le x\le 1,\ x^2\le y\le 1\}$

$D=\{(x,y) \mid 0\le y\le 1,\ 0\le x\le \sqrt{y}\}$.

View D as Type I region

$$\int_0^1\int_{x^2}^1 \sqrt{y}\sin y\,dy\,dx = \iint_D \sqrt{y}\sin y\,dA$$

$$= \int_0^1\int_0^{\sqrt{y}} \sqrt{y}\sin y\,dx\,dy$$

View D as Type II region.

$$= \int_0^1 (\sqrt{y}\sin y)\cdot x\Big|_{x=0}^{x=\sqrt{y}}\,dy$$

$$= \int_0^1 (\sqrt{y}\sin y)\sqrt{y}\,dy$$

$$= \int_0^1 y\sin y\,dy$$

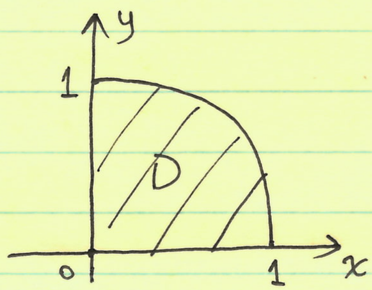
$$= y(-\cos y)\Big|_0^1 - \int_0^1 (-\cos y)\,dy$$

$$= -\cos(1) + \int_0^1 \cos y\,dy$$

$$= -\cos 1 + \sin y\Big|_0^1$$

$$= -\cos 1 + \sin 1.$$

3 (§15.4, #11)



The distance from a point (x,y) to the x-axis is $|y|$. Moreover, for $(x,y)\in D$, $y\geq 0$.

Thus, the density function is:

$$\rho(x,y) = c\cdot y, \quad \text{for } (x,y)\in D,$$

where $c>0$ is a proportional constant.

$$D = \{(x,y) \mid x^2+y^2\leq 1,\ x\geq 0,\ y\geq 0\}$$

$$= \{(r,\theta) \mid 0\leq r\leq 1,\ 0\leq\theta\leq\tfrac{\pi}{2}\},$$ as a polar rectangle.

The mass; $$m = \iint_D \rho(x,y)\,dA = \iint_D cy\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 c\cdot r\sin\theta\, r\,dr\,d\theta$$

$$= c\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^1 r^2\,dr$$

$$= c\left(-\cos\theta\Big|_0^{\frac{\pi}{2}}\right)\cdot\left(\frac{r^3}{3}\Big|_0^1\right)$$

$$= c\cdot 1\cdot\frac{1}{3}$$

$$= \frac{c}{3}.$$

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{c}\cdot\iint_D cxy\,dA$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\int_0^1 r^2\cos\theta\sin\theta\, r\,dr\,d\theta$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\int_0^1 r^3\,dr$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\,d\theta\int_0^1 r^3\,dr$$

$$= 3\cdot\left(-\frac{\cos(2\theta)}{4}\Big|_0^{\frac{\pi}{2}}\right)\cdot\left(\frac{r^4}{4}\Big|_0^1\right)$$

$$= 3\cdot\frac{1}{2}\cdot\frac{1}{4}$$

$$= \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{c}\cdot\iint_D cy^2\,dA$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\int_0^1 r^2\sin^2\theta\, r\,dr\,d\theta$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta\int_0^1 r^3\,dr$$

$$= 3\cdot\int_0^{\frac{\pi}{2}}\frac{1-\cos(2\theta)}{2}\,d\theta\int_0^1 r^3\,dr$$

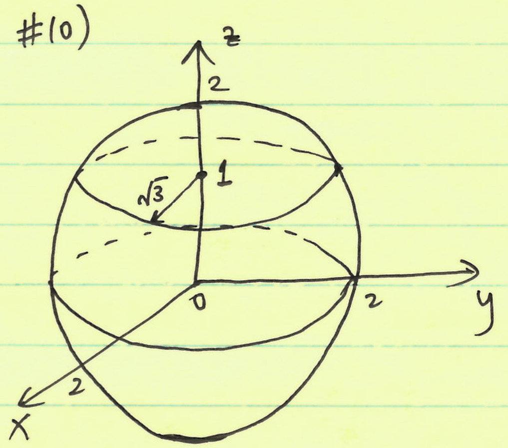
$$= 3\cdot\frac{1}{2}\left(\theta-\frac{\sin(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}}\cdot\left(\frac{r^4}{4}\Big|_0^1\right)$$

$$= 3\cdot\frac{\pi}{4}\cdot\frac{1}{4}$$

$$= \frac{3\pi}{16}.$$

Thus, the center of mass is $\left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

4. (§15.5, #10)



Set $z=1$ in $x^2+y^2+z^2=4$, we obtain $x^2+y^2=3$.

$$D=\{(x,y)\mid 0\le x^2+y^2\le 3\}=\{(r,\theta)\mid 0\le r\le\sqrt{3},\quad 0\le\theta\le 2\pi\}$$

polar rectangle.

The surface is $S=\{(x,y,z)\mid z=\sqrt{4-x^2-y^2},\quad 0\le x^2+y^2\le 3\}$.

$$A(S)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$=\iint_D\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\,dA$$

$$=\iint_D\frac{2}{\sqrt{4-x^2-y^2}}\,dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2}{\sqrt{4-r^2}}\cdot r\cdot dr\,d\theta=\int_0^{2\pi}d\theta\cdot\int_0^{\sqrt{3}}2\frac{r}{\sqrt{4-r^2}}\,dr$$

$$=2\pi\cdot 2\cdot\left(-\sqrt{4-r^2}\right)\Big|_0^{\sqrt{3}}$$

$$=2\pi\cdot 2\,(-1+2)$$

$$=4\pi.$$

5. (§15.3, #40)

a) D_a is a polar rectangle, $\{(r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$

Then, $\displaystyle\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2}\cdot r\,dr\,d\theta$

$$= \int_0^{2\pi} d\theta \int_0^a e^{-r^2}\cdot r\,dr$$

$$= 2\pi\cdot\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a$$

$$= 2\pi\left(-\frac{1}{2}e^{-a^2}+\frac{1}{2}\right)$$

$$= \pi(1-e^{-a})$$

Thus, $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA$

$$= \lim_{a\to\infty}\pi(1-e^{-a})$$

$$= \pi.$$

b) $\displaystyle\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA$

$\overset{\text{by (a)}}{\Longrightarrow}$ $\displaystyle\pi = \lim_{a\to\infty}\left[\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dy\,dx\right]$

$$= \lim_{a\to\infty}\left[\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}\cdot e^{-y^2}\,dy\,dx\right] = \lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}dx\int_{-a}^{a} e^{-y^2}dy\right]$$

$$= \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}dx\cdot\lim_{a\to\infty}\int_{-a}^{a} e^{-y^2}dy$$

Thus, $\int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy = \pi.$

c). $\pi = \int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy$

$= \int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-x^2}dx$

$\Rightarrow \pi = \left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2$, and since $e^{-x^2} \geq 0$ for all real x, we have:

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

d) Let $t = \sqrt{2} \cdot x$, $dt = \sqrt{2} \cdot dx$. $\left(x = \frac{t}{\sqrt{2}},\ dx = \frac{1}{\sqrt{2}}dt \right)$

Since, $\lim_{x \to \infty} t = \lim_{x \to \infty} \sqrt{2} \cdot x = \infty$, and $\lim_{x \to -\infty} t = \lim_{x \to -\infty} \sqrt{2} \cdot x = -\infty$, we have:

$\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt$

$\Rightarrow \sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt \Rightarrow \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \sqrt{2\pi}.$