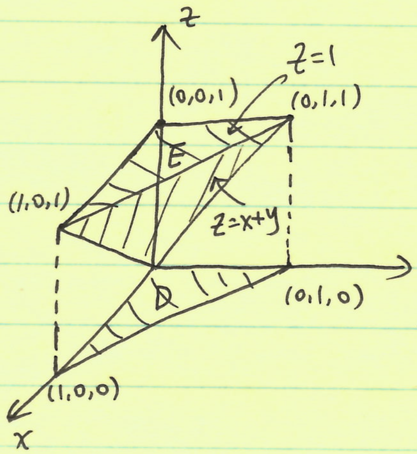
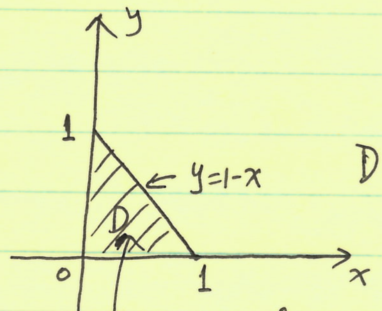
Homework 2. Solutions.

1. (§15.6, #16)





D is type I region.

The projection of E onto the xy-plane.

The equation of the plane through (0,0,0), (1,0,1) and (0,1,1) is

$$x+y-z=0 \quad (\Leftrightarrow z=x+y).$$

The solid tetrahedron can be described as:

$$T\cancel{E}=\{(x,y,z)\mid (x,y)\in D,\quad x+y\le z\le 1\} \leftarrow \text{Type 1}$$

$$=\{(x,y,z)\mid 0\le x\le 1,\quad 0\le y\le 1-x,\quad x+y\le z\le 1\}$$

$$\iiint_T xz\,dV = \int_0^1\int_0^{1-x}\int_{x+y}^1 xz\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x} x\,\frac{z^2}{2}\Big|_{z=x+y}^{z=1}\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\frac{1}{2}\left(x-x(x+y)^2\right)dy\,dx$$

$$=\int_0^1\int_0^{1-x}\frac{1}{2}\left(x-x^3-2x^2y-xy^2\right)dy\,dx$$

$$= \int_0^1 \frac{1}{2}\left(xy - x^3y - x^2y^2 - \frac{x}{3}y^3 \right)\Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{2}\left(x(1-x) - x^3(1-x) - x^2(1-x)^2 - \frac{x}{3}(1-x)^3 \right) dx$$

$$= \frac{1}{2}\int_0^1 \left(x - x^2 - x^3 + x^4 - x^2 + 2x^3 - x^4 - \frac{1}{3}x + x^2 - x^3 + \frac{x^4}{3} \right) dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x^4}{3} - x^2 + \frac{2}{3}x \right) dx$$

$$= \frac{1}{2}\left(\frac{x^5}{15} - \frac{x^3}{3} + \frac{x^2}{3} \right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{15} - \frac{1}{3} + \frac{1}{3} \right)$$

$$= \frac{1}{30}.$$

2. (§15.7, #20)

$$E = \{ (x,y,z) \mid (x,y) \in D, \quad 0 \le z \le y+4 \}$$

$$D = \{ (x,y) \mid 1 \le x^2+y^2 \le 16 \} = \{ (r,\theta) \mid 1 \le r \le 4 \}$$

Thus in tems of cylinderical coordinates,

$$E = \{ (r,\theta,z) \mid 1 \le r \le 4, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le r\sin\theta + 4 \}$$

$$\iiint_E (x-y)\, dV = \int_0^{2\pi} \int_1^4 \int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^4 r^2(\cos\theta - \sin\theta) z \Big|_0^{z=r\sin\theta+4} dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^4 r^2 (\cos\theta - \sin\theta)(r\sin\theta + 4)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^4 r^3 (\cos\theta \sin\theta - \sin^2\theta) + 4r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}(\cos\theta\sin\theta - \sin^2\theta) + \frac{4}{3} r^3 (\cos\theta - \sin\theta) \Big|_{r=1}^{r=4} d\theta$$

$$= \int_0^{2\pi} \frac{255}{4}(\cos\theta\sin\theta - \sin^2\theta) + \frac{252}{3}(\cos\theta - \sin\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{255}{4}\left(\cos\theta\sin\theta - \frac{1-\cos(2\theta)}{2}\right) + \frac{252}{3}(\cos\theta - \sin\theta)\, d\theta$$

$$= \frac{255}{4}\left(\sin^2\theta - \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + \frac{252}{3}(\sin\theta + \cos\theta) \Big|_0^{2\pi}$$

$$= -\frac{255}{4} \cdot \pi$$

3. (§15.8, #26)

In spherical coordinates (ρ, θ, ϕ),

$$E = \left\{ (\rho, \theta, \phi) \,\middle|\, 1 \le \rho \le 2,\quad 0 \le \theta \le 2\pi,\quad 0 \le \phi \le \frac{\pi}{4} \right\}.$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_1^2 \rho \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_1^2 \rho^3 \, d\rho$$

$$= (-\cos\phi)\Big|_0^{\frac{\pi}{4}} \cdot 2\pi \cdot \frac{\rho^4}{4}\Big|_1^2$$

$$= \left(-\frac{\sqrt{2}}{2} + 1\right) \cdot 2\pi \cdot \left(\frac{16}{4} - \frac{1}{4}\right)$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 2\pi \cdot \frac{15}{4}$$

$$= \frac{15}{2}\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \pi .$$

4. (§15.8, #22)

In spherical coordinates: (ρ, θ, ϕ)

$$E = \left\{ (\rho, \theta, \phi) \,\middle|\, 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3} \right\}$$

$$\iiint_E y^2 z^2 \, dV = \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^1 \rho^2 \sin^2\phi \sin^2\theta \cdot \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{3}} \sin^2\phi \cos^2\phi \sin\phi \, d\phi \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^1 \rho^6 \, d\rho$$

$$= \int_0^{\frac{\pi}{3}} (1 - \cos^2\phi) \cos^2\phi \sin\phi \, d\phi \quad \int_0^{2\pi} \frac{1 - \cos(2\theta)}{2} \, d\theta \int_0^1 \rho^6 \, d\rho$$

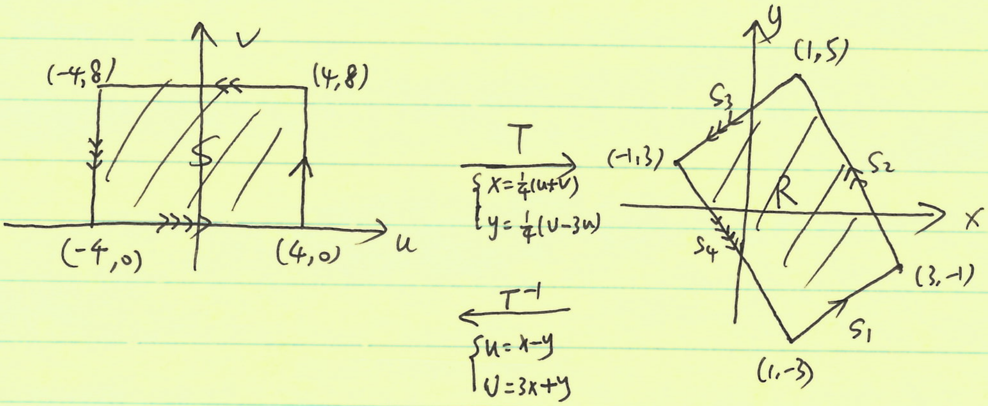
$$= \int_0^{\frac{\pi}{3}} (\cos^2\phi - \cos^4\phi) \sin\phi \, d\phi \cdot \left(\frac{1}{2}\theta - \frac{\sin(2\theta)}{4} \right) \Bigg|_0^{2\pi} \cdot \frac{\rho^7}{7} \Bigg|_0^1$$

$$= \left(-\frac{\cos^3\phi}{3} + \frac{\cos^5\phi}{5} \right) \Bigg|_0^{\frac{\pi}{3}} \cdot \pi \cdot \frac{1}{7}$$

$$= \left(-\frac{\frac{1}{8}}{3} + \frac{\frac{1}{32}}{5} - \left(-\frac{1}{3} + \frac{1}{5} \right) \right) \frac{\pi}{7}$$

$$= \frac{\pi}{7} \left(\frac{7}{24} - \frac{31}{160} \right)$$

5 (§15.9, #16)



T^{-1}:

(x,y)	(u,v)
$(-1,3)$	$(-4,0)$
$(1,5)$	$(-4,8)$
$(1,-3)$	$(4,0)$
$(3,-1)$	$(4,8)$

$S_1 = \{(x,y) \mid y = x-4,\quad -1 \le x \le 3\}$

$$\begin{cases} u = x-y = 4 \\ v = 4(x-1) \end{cases} \Rightarrow \begin{cases} u = 4 \\ 0 \le v = 4(x-1) \le 8 \end{cases}$$

$S_2 = \{(x,y) \mid y = -3x+8,\quad 1 \le x \le 3\}$

$$\begin{cases} u = x-y = 4(x-2) \\ v = 8 \end{cases} \Rightarrow \begin{cases} -4 \le u = 4(x-2) \le 4 \\ v = 8 \end{cases}$$

$$S_3 = \{(x,y) \mid y = x+4,\ -1 \le x \le 1\}$$

$$\begin{cases} u = x-y = -4 \\ v = 3x+y = 4(x+1) \end{cases} \Rightarrow \begin{cases} u = -4 \\ 0 \le v = 4(x+1) \le 8 \end{cases}$$

$$S_4 = \{(x,y) \mid y = -3x,\ -1 \le x \le 1\}$$

$$\begin{cases} u = x-y = 4x \\ v = 3x+y = 0 \end{cases} \Rightarrow \begin{cases} -4 \le u = 4x \le 4 \\ v = 0 \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

$$\iint_R (4x+8y)\,dA = \iint_S (u+v+2v-6u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^8 \int_{-4}^4 (3v-5u)\cdot\frac{1}{4}\cdot du\,dv$$

$$= \frac{1}{4}\int_0^8 \left(3vu - \frac{5}{2}u^2\right)\Big|_{-4}^{4} dv$$

$$= \frac{1}{4}\int_0^8 24v\,dv$$

$$= \int_0^8 6v\,dv$$

$$= 3v^2\Big|_0^8 = 3\cdot 64 = 192.$$