Homework 3. Solutions.

1. (a) (§16.2, #2).

$$\int_C \frac{x}{y}\,ds = \int_1^2 \frac{t^3}{t^4}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^2 \frac{1}{t}\sqrt{(3t^2)^2+(4t^3)^2}\,dt$$

$$= \int_1^2 \frac{1}{t}\sqrt{9t^4+16t^6}\,dt$$

$$= \int_1^2 \frac{1}{t}\,t^2\sqrt{9+16t^2}\,dt$$

$$= \int_1^2 t\sqrt{9+16t^2}\,dt$$

$$\underset{du=32t}{\overset{u=9+16t^2}{=\!=\!=}} \int_{25}^{73} \sqrt{u}\,\frac{1}{32}\,du = \frac{1}{32}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{25}^{73} = \frac{1}{48}\left(73^{\frac{3}{2}} - 25^{\frac{3}{2}}\right)$$

$$\cancel{= \frac{1}{32}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{5}^{\sqrt{73}}}$$

$$\cancel{= \frac{1}{48}\left(73^{\frac{3}{4}} - 5^{\frac{3}{2}}\right)}$$

(b) (§16, #10) Parametrize the line segment C as:

$$\vec{r}(t) = (1-t)(3,1,2) + t(1,2,5)$$

$$= (3-3t,\ 1-t,\ 2-2t) + (t,\ 2t,\ 5t)$$

$$= (3-2t,\ 1+t,\ 2+3t), \qquad 0\le t\le 1$$

i.e. $x(t)=3-2t,\quad y(t)=1+t,\quad z(t)=2+3t,\quad 0\le t\le 1.$

$$\int_C y^2 z\,ds = \int_0^1 (1+t)^2\cdot(2+3t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^1 (1+2t+t^2)(2+3t)\sqrt{(-2)^2+(1)^2+(3)^2}\, dt$$

$$= \int_0^1 (2+3t+4t+6t^2+2t^2+3t^3)\sqrt{4+1+9}\, dt$$

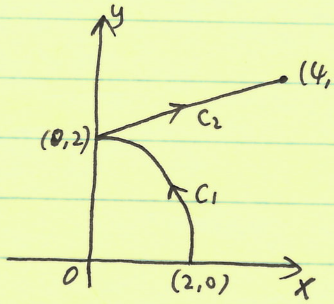
$$= \sqrt{14}\int_0^1 (3t^3+8t^2+7t+2)\, dt$$

$$= \sqrt{14}\left(\frac{3}{4}t^4+\frac{8}{3}t^3+\frac{7}{2}t^2+2t\right)\Big|_0^1$$

$$= \sqrt{14}\left(\frac{3}{4}+\frac{8}{3}+\frac{7}{2}+2\right)$$

$$= \sqrt{14}\cdot\frac{107}{12}.$$

2. (§16.2, #8)



Parametrization:

$$C_1: \begin{cases} x(t) = 2\cos t, \\ y(t) = 2\sin t, \end{cases} \quad 0 \le t \le \frac{\pi}{2},$$

$$C_2: \vec{r}(t) = (1-t)(0,2) + t(4,3)$$
$$= (0, 2-2t) + (4t, 3t)$$
$$= (4t, 2+t), \quad 0 \le t \le 1$$

i.e.
$$\begin{cases} x(t) = 4t \\ y(t) = 2+t \end{cases}, \quad 0 \le t \le 1,$$

$$\int_{C_1} x^2dx + y^2dy = \int_0^{\frac{\pi}{2}} (2\cos t)^2\cdot(-2\sin t)\,dt + (2\sin t)^2\cdot(2\cos t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} -8\cos^2 t\sin t\,dt + 8\sin^2 t\cos t\,dt$$

$$= 8\cdot\frac{1}{3}\cdot\cos^3 t\Big|_0^{\frac{\pi}{2}} + 8\cdot\frac{1}{3}\cdot\sin^3 t\Big|_0^{\frac{\pi}{2}}$$

$$= 0 - 8\cdot\frac{1}{3}\cdot 1 + 8\cdot\frac{1}{3}\cdot 1 - 0 = 0.$$

$$\int_{C_2} x^2dx + y^2dy = \int_0^1 (4t)^2 \cdot 4dt + (2+t)^2 \cdot dt$$

$$= \int_0^1 \left(64t^2 + (2+t)^2\right)dt$$

$$= \frac{64}{3}t^3 + \frac{1}{3}(2+t)^3 \Big|_0^1$$

$$= \frac{64}{3} + \frac{1}{3}\left(3^3 - 2^3\right)$$

$$= \frac{64 + 27 - 8}{3}$$

$$= \frac{83}{3}.$$

Thus, $\int_C x^2dx + y^2dy = \int_{C_1} x^2dx + y^2dy + \int_{C_2} x^2dx + y^2dy = 0 + \frac{83}{3} = \frac{83}{3}.$

3. (a) (§16.2, #19).

$$\vec{r}'(t) = 3t^2 \cdot \vec{i} + 2t \cdot \vec{j}$$

$$\vec{F}(\vec{r}(t)) = t^3 \cdot (t^2)^2 \cdot \vec{i} - (t^3)^2 \cdot \vec{j} = t^7 \cdot \vec{i} - t^6 \cdot \vec{j}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^1 \left(3t^9 - 2t^7\right)dt$$

$$= \frac{3}{10}t^{10} - \frac{2}{8} \cdot t^8 \Big|_0^1$$

$$= \frac{3}{10} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20}.$$

(6) (§16.2, # 22).

$$\vec{r}(t) = (\cos t)\cdot\vec{i} + (\sin t)\cdot\vec{j} + t\cdot\vec{k}$$

$$\vec{r}\,'(t) = (-\sin t)\cdot\vec{i} + (\cos t)\cdot\vec{j} + \vec{k}$$

$$\vec{F}(\vec{r}(t)) = (\cos t)\cdot\vec{i} + (\sin t)\cdot\vec{j} + (\sin t\cdot\cos t)\cdot\vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\cdot dt$$

$$= \int_0^{\pi} -(\sin t)(\cos t) + \sin t\cdot\cos t + \sin t\cdot\cos t\ dt$$

$$= \int_0^{\pi} \cos t\cdot\sin t\ dt$$

$$= \frac{1}{2}(\sin t)^2\Big|_0^{\pi}$$

$$= 0 - 0$$

$$= 0.$$

4. (§16.3, #6). $\vec{F}(x,y) = (ye^x)\,\vec{i} + (e^x+e^y)\,\vec{j}.$

$\Rightarrow\ P(x,y) = ye^x, \quad Q(x,y) = e^x + e^y.$

- $\frac{\partial P}{\partial y} = e^x, \quad \frac{\partial Q}{\partial x} = e^x \Rightarrow \frac{\partial P}{\partial y} = e^x = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}(x,y)$ is a conservative vector field.

- $\begin{cases} f_x = ye^x \\ f_y = e^x + e^y \end{cases}$ $f_x = ye^x \Rightarrow f(x,y) = \int ye^x dx = ye^x + c(y) \Rightarrow f_y = e^x + c'(y)$

$\Rightarrow\ e^x + e^y = f_y = e^x + c'(y) \Rightarrow c'(y) = e^y \Rightarrow c(y) = e^y + c.$

$\Rightarrow \quad f(x) = ye^x + c(y) = ye^x + e^y + c$, where c is an arbitrary constant.

Thus, $f(x,y) = ye^x + e^y$ is a function s.t. $\nabla f = \vec{F}$

Just set $c=0$. of course, you can choose any value for c to get an answer.

5. (§16.3, #14) (a) $\vec{F} = \nabla f \Rightarrow \begin{cases} f_x = (1+xy)e^{xy} \\ f_y = x^2 e^{xy} \end{cases}$

$$f_x = (1+xy)e^{xy} \Rightarrow f(x,y) = \int (1+xy)e^{xy}\,dx$$

$$= \int e^{xy} + xye^{xy}\,dx$$

$$= \int e^{xy}dx + \int xye^{xy}dx$$

$$= \frac{1}{y}e^{xy} + \int x\,(e^{xy})'dx$$

$$= \frac{1}{y}e^{xy} + xe^{xy} - \int e^{xy}dx$$

$$= \frac{1}{y}e^{xy} + xe^{xy} - \frac{1}{y}e^{xy} + c(y)$$ a function of y

$$= xe^{xy} + c(y)$$

$$\Rightarrow f_y(x,y) = x^2e^{xy} + c'(y)$$

$$\xrightarrow{\text{with } f_y = x^2e^{xy}} \quad x^2e^{xy} + c'(y) = f_y(x,y) = x^2e^{xy} \Rightarrow c'(y) = 0 \Rightarrow c(y) = c \leftarrow \text{a constant.}$$

$\Rightarrow f(x,y) = xe^{xy} + C(y) = xe^{xy} + c$, where c is an arbitrary constant.

Thus, $f(x,y) = xe^{xy}$ is a function such that $\nabla f = \vec{F}$.

(b) $\vec{r}(t) = (\cos t)\vec{i} + (2\sin t)\vec{j}$, $\quad 0 \le t \le \frac{\pi}{2}$.

$\vec{r}(0) = (1,0)$, $\quad \vec{r}(\frac{\pi}{2}) = (0,2)$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,2) - f(1,0) = 0 - 1 = -1.$