Homework 4. Solutions.

1. (§16.3, #18)

(a) $\vec{F} = \nabla f \iff \begin{cases} \dfrac{\partial f}{\partial x} = \sin y \\ \dfrac{\partial f}{\partial y} = x\cos y + \cos z \\ \dfrac{\partial f}{\partial z} = -y\sin(z) \end{cases}$

- $\dfrac{\partial f}{\partial x} = \sin y \Rightarrow f(x,y,z) = \displaystyle\int \frac{\partial f}{\partial x}\,dx = x\sin y + h(y,z)$

  i.e. $f(x,y,z) = x\sin y + h(y,z)$

- $\left.\begin{matrix} \dfrac{\partial f}{\partial y} = x\cos y + \dfrac{\partial h}{\partial y}(y,z) \\ \| \\ x\cos y + \cos z \end{matrix}\right\} \Rightarrow \dfrac{\partial h}{\partial y}(y,z) = \cos z$

  $\Rightarrow h(y,z) = y\cos z + g(z)$

  i.e. $f(x,y,z) = x\sin y + y\cos z + g(z)$

- $-y\sin z = \dfrac{\partial f}{\partial z} = -y\sin z + g'(z) \Rightarrow g'(z) = 0 \Rightarrow g(z) = c$

Thus, $f(x,y,z) = x\sin y + y\cos z + c$, where c is an arbitrary constant.

So $f(x,y,z) = x\sin y + y\cos z$ is a potential function such that $\vec{F} = \nabla f$.

(b). $\vec{r}(0) = (0,0,0)$, $\vec{r}(\frac{\pi}{2}) = (1, \frac{\pi}{2}, \pi)$.

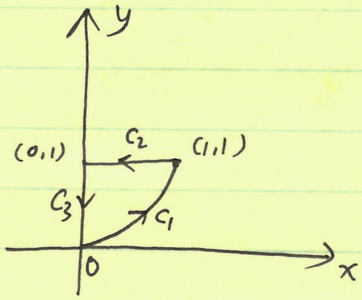
Thus, $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(1,\tfrac{\pi}{2},\pi) - f(0,0,0)$

$= 1 + \frac{\pi}{2}\cos\pi - 0$

$= 1 - \frac{\pi}{2}$

2. (§16.4, #4)

(a)



$C_1: \begin{cases} x=t, & 0 \le t \le 1. \\ y=t^2, \end{cases}$

$C_2: \begin{cases} x=1-t, & 0 \le t \le 1 \\ y=1 \end{cases}$

$C_3: \begin{cases} x=0 \\ y=1-t, & 0 \le t \le 1. \end{cases}$

$$\oint_C x^2y^2dx + xy\,dy = \int_{C_1} x^2y^2dx + xy\,dy + \int_{C_2} x^2y^2dx + xy\,dy + \int_{C_3} x^2y^2dx + xy\,dy$$

$$= \int_0^1 t^2(t^2)^2dt + t\cdot t^2\cdot 2t\,dt + \int_0^1 (1-t)^2(-dt) + \int_0^1 0\,(-dt)$$

$$= \int_0^1 t^6 + 2t^4 - (1-t)^2\,dt$$

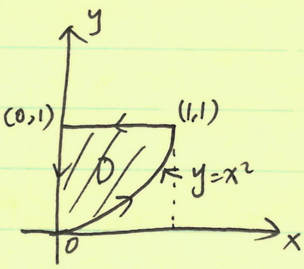
$$= \int_0^1 t^6 + 2t^4 - 1 + 2t - t^2\,dt$$

$$= \frac{t^7}{7} + \frac{2}{5}t^5 - t + t^2 - \frac{t^3}{3}\Big|_0^1$$

$$= \frac{1}{7} + \frac{2}{5} - 1 + 1 - \frac{1}{3}$$

$$= \frac{15+42-35}{105} = \frac{57-35}{105} = \frac{22}{105}.$$

(b)



$$\oint_C x^2y^2\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2y^2)\right) dA$$

$$D = \{(x,y) \mid 0 \le x \le 1, \quad x^2 \le y \le 1\}.$$

$$= \iint_D (y - 2x^2y)\, dA$$

$$= \int_0^1 \int_{x^2}^1 y - 2x^2y \, dy\, dx$$

$$= \int_0^1 (1-2x^2)\frac{y^2}{2}\Big|_{y=x^2}^{y=1} dx$$

$$= \int_0^1 (1-2x^2)\frac{1}{2}(1-x^4)\, dx$$

$$= \frac{1}{2}\int_0^1 (1 - x^4 - 2x^2 + 2x^6)\, dx$$

$$= \frac{1}{2}\left(x - \frac{x^5}{5} - \frac{2}{3}x^3 + \frac{2}{7}x^7\right)\Big|_0^1$$

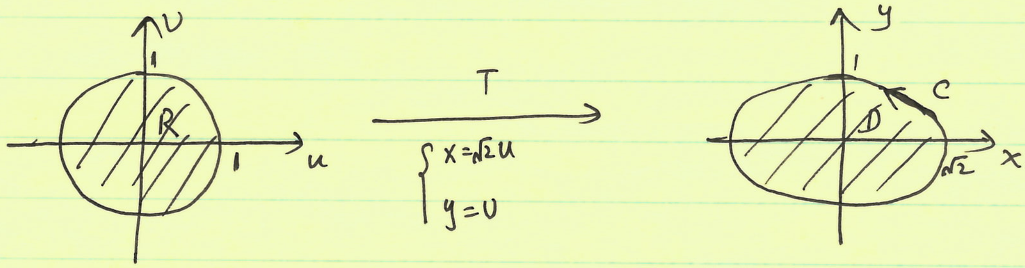
$$= \frac{1}{2}\left(1 - \frac{1}{5} - \frac{2}{3} + \frac{2}{7}\right)$$

$$= \frac{1}{2}\cdot\frac{105 - 21 - 70 + 30}{105}$$

$$= \frac{1}{2}\cdot\frac{44}{105}$$

$$= \frac{22}{105}.$$

3. (§16.5, #8)



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{2}$$

$$\int_C y^4\,dx + 2xy^3\,dy \overset{\text{Green's Thm}}{=\!=} \iint_D \frac{\partial}{\partial x}(2xy^3) - \frac{\partial}{\partial y}(y^4)\;dA$$

$$= \iint_D 2y^3 - 4y^3\,dA$$

$$= -2\iint_D y^3\,dA$$

Use transformation T. given as above picture

$$= -2\iint_R v^3\cdot\sqrt{2}\,dA$$

Use polar coordinates $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$

$$= -2\int_0^{2\pi}\int_0^1 \sqrt{2}\cdot(r\sin\theta)^3\cdot r\,dr\,d\theta$$

$$= -2\sqrt{2}\int_0^{2\pi}\sin^3\theta\,d\theta\int_0^1 r^4\,dr$$

$$= -2\sqrt{2}\int_0^{2\pi}(1-\cos^2\theta)\sin\theta\,d\theta\int_0^1 r^4\,dr$$

$$\overset{\substack{w=\cos\theta \\ dw=-\sin\theta\,d\theta}}{=\!=} -2\sqrt{2}\underbrace{\left(\int_1^1 (1-w^2)(-dw)\right)}_{\shortparallel\atop 0}\left(\frac{r^5}{5}\Big|_0^1\right)$$

$$= 0.$$

4. (a) (§16.5, #14).

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^4 & x^2z^4 & 4x^2yz^3 \end{vmatrix}$$

$$= \vec{i}\left(4x^2z^3 - 4x^2z^3\right) - \vec{j}\left(8xyz^3 - 4xyz^3\right) + \vec{k}\left(2xz^4 - xz^4\right)$$

$$= -4xyz^3\vec{j} + xz^4\vec{k}.$$

$$\neq \vec{0}.$$

So $\vec{F}$ is not conservative, since $\vec{F}$ is defined everywhere.

(b) (§16.5, #18)

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin(yz) & ze^x\cos(yz) & ye^x\cos(yz) \end{vmatrix}$$

$$= \vec{i}\left(e^x\cos(yz) - ye^xz\sin(yz) - e^x\cos(yz) + ze^xy\cos(yz)\right)$$

$$- \vec{j}\left(ye^x\cos(yz) - e^xy\cos(yz)\right)$$

$$+ \vec{k}\left(ze^x\cos(yz) - e^xz\cos(yz)\right)$$

$$= \vec{0}.$$

$\vec{F}$ is defined everywhere.

Thus, $\vec{F}$ is conservative.

- $\vec{F} = \nabla f \iff \begin{cases} \frac{\partial f}{\partial x} = e^x \sin(yz) \\ \frac{\partial f}{\partial y} = ze^x \cos(yz) \\ \frac{\partial f}{\partial z} = ye^x \cos(yz) \end{cases}$

- $\frac{\partial f}{\partial x} = e^x \sin(yz) \Rightarrow \quad f(x,y,z) = \int \frac{\partial f}{\partial x} dx = \int e^x \sin(yz) dx$

$$= e^x \sin(yz) + h(y,z)$$

i.e. $f(x,y,z) = e^x \sin(yz) + h(y,z)$

- $\left.\begin{array}{l} \frac{\partial f}{\partial y} = e^x z \cos(yz) + \frac{\partial h}{\partial y}(y,z) \\ \quad \parallel \\ z e^x \cos(yz) \end{array}\right\} \Rightarrow \quad \frac{\partial h}{\partial y}(y,z) = 0$

$$\Rightarrow \quad h(y,z) = g(z).$$

i.e. $f(x,y,z) = e^x \sin(yz) + g(z)$

- $ye^x \cos(yz) = \frac{\partial f}{\partial z} = e^x y \cos(yz) + g'(z) \Rightarrow g'(z) = 0 \Rightarrow g(z) = C.$

Thus, $f(x,y,z) = e^x \sin(yz) + C$, where C is an arbitrary constant.

So $f(x,y,z) = e^x \sin(yz)$ is a potential function such that $\vec{F} = \nabla f$.