Homework 5. Solutions.

1. (§16.6, #36). $\vec{r}(u,v) = (\sin u,\ \cos u \sin v,\ \sin v)$

$$\vec{r}_u = (\cos u,\ -\sin u \sin v,\ 0)$$

$$\vec{r}_v = (0,\ \cos u \cos v,\ \cos v)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u & -\sin u \sin v & 0 \\ 0 & \cos u \cos v & \cos v \end{vmatrix}$$

$$= (-\sin u \sin v \cos v,\ -\cos u \cos v,\ \cos^2 u \cos v)$$

$$\vec{r}\left(\tfrac{\pi}{6}, \tfrac{\pi}{6}\right) = \left(\sin\left(\tfrac{\pi}{6}\right),\ \cos\left(\tfrac{\pi}{6}\right)\cdot\sin\left(\tfrac{\pi}{6}\right),\ \sin\left(\tfrac{\pi}{6}\right)\right)$$

$$= \left(\tfrac{1}{2},\ \tfrac{\sqrt{3}}{4},\ \tfrac{1}{2}\right)$$

$$(\vec{r}_u \times \vec{r}_v)\left(\tfrac{\pi}{6}, \tfrac{\pi}{6}\right) = \left(-\sin\left(\tfrac{\pi}{6}\right)\sin\left(\tfrac{\pi}{6}\right)\cos\left(\tfrac{\pi}{6}\right),\ -\cos\left(\tfrac{\pi}{6}\right)\cos\left(\tfrac{\pi}{6}\right),\ \cos^2\left(\tfrac{\pi}{6}\right)\cdot\cos\left(\tfrac{\pi}{6}\right)\right)$$

$$= \left(-\tfrac{\sqrt{3}}{8},\ -\tfrac{3}{4},\ \tfrac{3\sqrt{3}}{8}\right)$$

Thus $-\frac{\sqrt{3}}{8}\left(x - \frac{1}{2}\right) - \frac{3}{4}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z - \frac{1}{2}\right) = 0$ is an eq^{n} of the tangent plane.

2. (§16.7, #8) $\vec{r}(u,v) = (2uv,\ u^2-v^2,\ u^2+v^2)$, $u^2+v^2 \le 1$.

$\vec{r}_u = (2v,\ 2u,\ 2u)$

$\vec{r}_v = (2u,\ -2v,\ 2v)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2v & 2u & 2u \\ 2u & -2v & 2v \end{vmatrix} = (4uv+4uv,\ -4v^2+4u^2,\ -4v^2-4u^2)$$

$$= (8uv,\ 4u^2-4v^2,\ -4u^2-4v^2)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(8uv)^2 + (4u^2-4v^2)^2 + (-4u^2-4v^2)^2}$$

$$= \sqrt{64u^2v^2 + 16u^4 - 32u^2v^2 + 16v^4 + 16u^4 + 32u^2v^2 + 16v^4}$$

$$= \sqrt{32u^4 + 64u^2v^2 + 32v^4}$$

$$= \sqrt{32\,(u^2+v^2)^2}$$

$$= \sqrt{32}\,(u^2+v^2).$$

$$\iint_S (x^2+y^2)\,dS = \iint_D \left((2uv)^2 + (u^2-v^2)^2\right) \cdot |\vec{r}_u \times \vec{r}_v| \cdot dA \qquad \left(D = \{(u,v) \mid u^2+v^2 \le 1\}\right)$$

$$= \iint_D (4u^2v^2 + u^4 - 2u^2v^2 + v^4) \cdot \sqrt{32}\,(u^2+v^2)\,dA$$

$$= \iint_D \sqrt{32}\,(u^2+v^2)^3\,dA = \int_0^{2\pi}\int_0^1 \sqrt{32}\cdot(r^2)^3 \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{32}\,r^7\,dr = 2\pi \cdot \sqrt{32} \cdot \frac{1}{8} = \sqrt{2}\cdot\pi.$$

3. (§16.7, #24). First, find a parametrization for the surface S:

$$\vec{r}(x,y) = (x, y, \sqrt{x^2+y^2}), \qquad 1 \le x^2+y^2 \le 9.$$

$D = \{(x,y) \mid 1 \le x^2+y^2 \le 9\}$ ← domain for parameters.

$$\vec{r}_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right), \quad \vec{r}_y = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)$$

This is an <u>upward</u> normal vector, since $z=1>0$.

So we need $\vec{r}_y \times \vec{r}_x = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)$, because in the problem the surface S is oriented <u>downward</u>.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_y \times \vec{r}_x)\, dA = \iint_D (-x, -y, (\sqrt{x^2+y^2})^3) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right) dA$$

$$= \iint_D \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} - (x^2+y^2)^{\frac{3}{2}}\right) dA$$

$$= \iint_D \left(-(x^2+y^2)^{\frac{1}{2}} - (x^2+y^2)^{\frac{3}{2}}\right) dA$$

$$= \int_0^{2\pi} \int_{\cancel{0}\,1}^{\cancel{1}\,3} (-r - r^3)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\cancel{0}\,1}^{\cancel{1}\,3} -r^2 - r^4\, dr = 2\pi \cdot \left(-\frac{r^3}{3} - \frac{r^5}{5}\right)\Bigg|_{\cancel{0}\,1}^{\cancel{1}\,3}$$

$$\cancel{= 2\pi\left(-\frac{1}{3} - \frac{1}{5}\right) = -\frac{16}{15}\pi.}$$

$$= 2\pi\left\{\left[-3^2 - \frac{3^5}{5}\right] - \left[-\frac{1}{3} - \frac{1}{5}\right]\right\} = 2\pi\left\{-9 - \frac{3^5}{5} + \frac{8}{15}\right\}$$

4. (§16.8, #2) The boundary of the surface S is given by:

$$\begin{cases} z = 1 - x^2 - y^2 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$$

thus, a parametrization of ∂S is given by:

$$\vec{r}(\theta) = (\cos\theta, \sin\theta, 0), \quad 0 \le \theta \le 2\pi .$$

This parametrization gives the counterclockwise orientation for the boundary curve. And because the induced orientation for the boundary ∂S from the given (upward) orientation of the surface S, $\vec{r}(\theta) = (\cos\theta, \sin\theta, 0)$, $0 \le \theta \le 2\pi$ is a parametric equation that we need.

Then, by Stokes' Theorem:

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} \underset{\text{Stokes}}{=} \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta)\, d\theta = \int_0^{2\pi} (0, \sin^2\theta, \cos\theta\sin\theta) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta$$

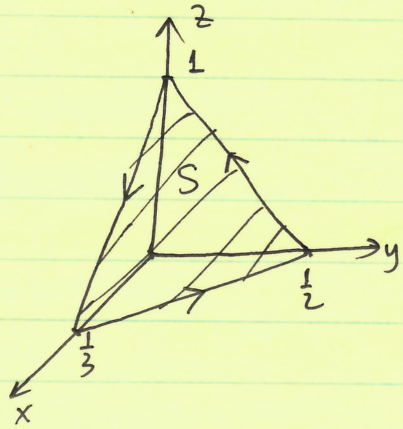
this means the boundary curve of S with induced orientation from the given orientation of S

$$= \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta$$

$$= \frac{1}{3}\sin^3\theta \Big|_0^{2\pi}$$

$$= 0 .$$

5. (§16.8, #8)



S is the part of the plane $3x+2y+z=1$ in the first octant.

A parametrization of S is :

$$\vec{r}(x,y) = (x,\ y,\ 1-3x-2y), \quad (x,y)\in D,$$

where $D = \{(x,y) \mid 0 \le x \le \frac{1}{3},\ 0 \le y \le \frac{1}{2} - \frac{3}{2}x\}$.

$\vec{r}_x = (1, 0, -3)$, $\quad \vec{r}_y = (0, 1, -2)$,

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = (3, 2, 1)$$ ← upward normal vector of S, this matches the given orientation of the boundary curve C.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = (x-y, -y, 1)$$

By Stokes' Theorem :

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S} = \iint_D (\text{curl}\,\vec{F})(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y)\, dA$$

$$= \iint_D (x-y, -y, 1)\cdot(3, 2, 1)\, dA$$

$$= \iint_D 3x-3y-2y+1\ dA$$

$$= \int_0^{\frac{1}{3}} \int_0^{\frac{1}{2}-\frac{3}{2}x} 3x-5y+1\, dy\, dx$$

$$=\int_0^{\frac{1}{3}} 3xy-\frac{5}{2}y^2+y\Big|_0^{\frac{1}{2}-\frac{3}{2}x}\, dx$$

$$=\int_0^{\frac{1}{3}} 3x\left(\frac{1}{2}-\frac{3}{2}x\right)-\frac{5}{2}\left(\frac{1}{2}-\frac{3}{2}x\right)^2+\left(\frac{1}{2}-\frac{3}{2}x\right) dx$$

$$=\int_0^{\frac{1}{3}}\left(\frac{1}{2}-\frac{3}{2}x\right)\left[3x-\frac{5}{4}+\frac{15}{4}x+1\right] dx$$

$$=\int_0^{\frac{1}{3}}\left(\frac{1}{2}-\frac{3}{2}x\right)\left(\frac{27}{4}x-\frac{1}{4}\right) dx$$

$$=\int_0^{\frac{1}{3}} -\frac{81}{8}x^2+\frac{27}{8}x+\frac{3}{8}x-\frac{1}{8}\, dx$$

$$=\int_0^{\frac{1}{3}} -\frac{81}{8}x^2+\frac{30}{8}x-\frac{1}{8}\, dx$$

$$=-\frac{27}{8}x^3+\frac{15}{8}x^2-\frac{1}{8}x\Big|_0^{\frac{1}{3}}$$

$$=-\frac{27}{8}\cdot\frac{1}{27}+\frac{15}{8}\cdot\frac{1}{9}-\frac{1}{8}\cdot\frac{1}{3}$$

$$=-\frac{1}{8}+\frac{5}{24}-\frac{1}{24}$$

$$=-\frac{1}{8}+\frac{4}{24}$$

$$=-\frac{1}{8}+\frac{1}{6}$$

$$=\frac{-3+4}{24}$$

$$=\frac{1}{24}.$$