Math 2X03 - Homework 1

Due: May 10, 2018 (by 10:00 pm)

(The following problems are from the textbook.)

1. (§15.2 #36) By subtracting two volumes, find the volume of the solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$.

2. (§15.2 #52) Evaluate the integral

$$\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y dy dx$$

   by reversing the order of integration.

3. (§15.4 #11) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

4. (§15.5 #10) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.

5. (§15.3 #40)

   (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$\begin{aligned} I &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx \\ &= \lim_{a \to \infty} \iint_{D_a} e^{-(x^2+y^2)} dA \end{aligned}$$

   where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dA = \pi.$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)} dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi.$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

(d) By making the change of variable $t = \sqrt{2}x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}.$$

(This is a fundamental result for probability and statistics.)