MATH 2X03 Midterm I

Dr. Changliang Wang

Date: May 15, 2018.

Duration of Exam: 1 hour

Solutions.

Family Name: ________________ **First Name:** ________________

Student Number: ________________________________

This examination paper contains 4 questions on a total of 6 pages. You are responsible for ensuring that your copy of the exam paper is complete. Please bring any discrepancies to the attention of your invigilator.

INSTRUCTIONS:

1. Answer all questions in the spaces provided, and make sure that you justify clearly all steps in your solutions.
2. The only calculator that is permitted in this examination is the Casio FX 991MS or MS Plus calculator.

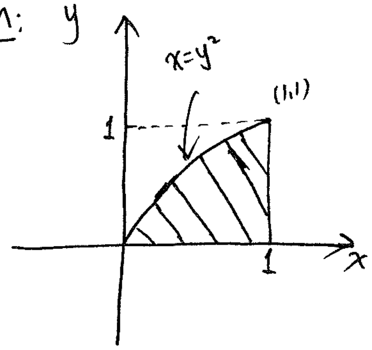
Question	Max Marks	Marks
1	4	
2	3	
3	4	
4	4	
Total	15	

Question 1. [4 marks] For the iterated integral

$$\int_0^1 \left(\int_{y^2}^1 y \cos(x^2) dx \right) dy$$

sketch the region of integration. Then evaluate the integral by reversing the order of integration.

Solution:



$$\int_0^1 \left(\int_{y^2}^1 y \cos(x^2)\, dx \right) dy$$

$$= \int_0^1 \left(\int_0^{\sqrt{x}} y \cos(x^2)\, dy \right) dx$$

$$= \int_0^1 \frac{y^2}{2} \cos(x^2) \Big|_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \frac{x}{2} \cos(x^2)\, dx$$

$$= \frac{1}{4} \sin(x^2) \Big|_0^1$$

$$= \frac{1}{4} \sin(1).$$

Question 2. [3 marks] Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies within the cylinder $x^2 + y^2 = 1$.

Solution:

$$D = \{(x,y) \mid x^2+y^2 \le 1\} = \{(r,\theta) \mid 0 \le r \le 1\}$$

$$\text{Surface area} = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA$$

$$= \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \int_0^1 \sqrt{1+4r^2}\, r\, dr$$

$$\overset{u=1+4r^2}{\underset{du=8r\,dr}{=\!=}} 2\pi \int_1^5 \sqrt{u}\cdot\frac{1}{8}\, du$$

$$= \frac{\pi}{4}\int_1^5 \sqrt{u}\, du$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}\Big|_1^5$$

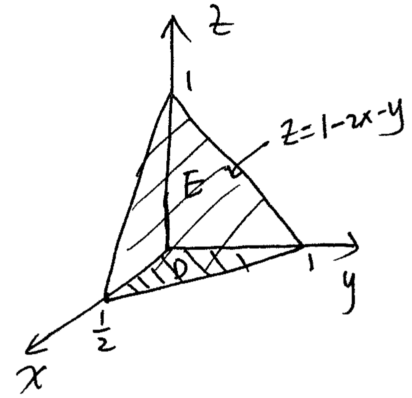
$$= \frac{\pi}{6}\left(5^{\frac{3}{2}}-1\right).$$

Question 3. [4 marks] Evaluate the triple integral

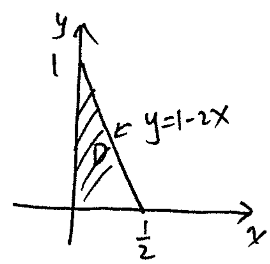
$$\iiint_E 2dV$$

where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 1$.

Solution:



base D:



$$E = \{(x,y,z) \mid 0 \le x \le \tfrac{1}{2}, \quad 0 \le y \le 1-2x, \quad 0 \le z \le 1-2x-y\}$$

$$\iiint_E 2\,dV = \int_0^{\frac{1}{2}} \int_0^{1-2x} \int_0^{1-2x-y} 2\,dz\,dy\,dx$$

$$= \int_0^{\frac{1}{2}} \int_0^{1-2x} 2z \Big|_{z=0}^{z=1-2x-y} dy\,dx$$

$$= \int_0^{\frac{1}{2}} \int_0^{1-2x} 2(1-2x) - 2y \; dy\,dx$$

$$= \int_0^{\frac{1}{2}} 2(1-2x)y - y^2 \Big|_{y=0}^{y=(1-2x)} dx$$

$$= \int_0^{\frac{1}{2}} 2(1-2x)^2 - (1-2x)^2 \, dx$$

$$= \int_0^{\frac{1}{2}} (1-2x)^2 \, dx$$

$$= -\frac{1}{6}(1-2x)^3 \Big|_0^{\frac{1}{2}} = 0 - \left(-\frac{1}{6}\right) = \frac{1}{6}.$$

Question 4. [4 marks] Evaluate the triple integral

$$\iiint_E z\,dV$$

where E is the solid region above the cone $z = \sqrt{x^2 + y^2}$, below the sphere $x^2 + y^2 + z^2 = 1$, and in the first octant.

Solution: In the spherical coordinates:

$$E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le \phi \le \tfrac{\pi}{4}\}$$

$$\iiint_E z\,dV = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho\cos\phi\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \sin\phi\cos\phi\, d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 \rho^3 d\rho$$

$$= \left(\frac{1}{2}(\sin\phi)^2 \Big|_0^{\frac{\pi}{4}}\right) \cdot \frac{\pi}{2} \cdot \left(\frac{1}{4}\rho^4 \Big|_0^1\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2 \cdot \frac{\pi}{2} \cdot \frac{1}{4}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{1}{4}$$

$$= \frac{\pi}{32}.$$