MATH 2X03 Midterm II

Dr. Changliang Wang

Date: May 29, 2018.

Solutions.

Duration of Exam: 1 hour

Family Name: ____________ **First Name:** ____________

Student Number: ________________________

This examination paper contains 4 questions (with parts) on a total of 7 pages. You are responsible for ensuring that your copy of the exam paper is complete. Please bring any discrepancies to the attention of your invigilator.

INSTRUCTIONS:

1. Answer all questions in the spaces provided, and make sure that you justify clearly all steps in your solutions.
2. The only calculator that is permitted in this examination is the Casio FX 991MS or MS Plus calculator.

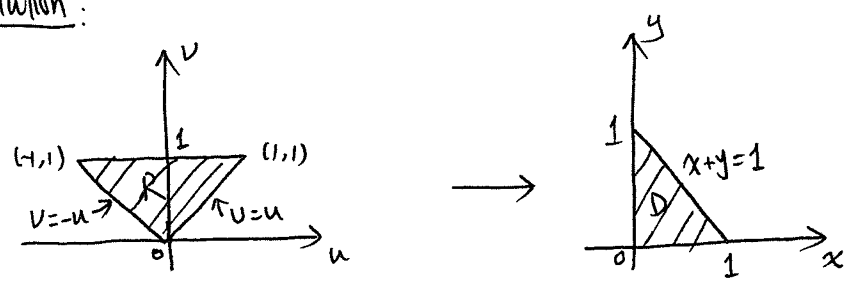
Question	Max Marks	Marks
1	3	
2	3	
3	5	
4	4	
Total	15	

Question 1. [3 marks] Use the transformation given by $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ to evaluate the double integral

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dA,$$

where D is the region bounded by lines $x = 0$, $y = 0$, and $x + y = 1$.

Solution:



$$\begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases} \Longleftrightarrow \begin{cases} u = x-y \\ v = x+y \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$R = \{(u,v) \mid 0 \le v \le 1, \ -v \le u \le v\}$.

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dA = \iint_R \cos\left(\frac{u}{v}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= \int_0^1 \int_{-v}^{v} \frac{1}{2} \cos\left(\frac{u}{v}\right) du\, dv$$

$$= \frac{1}{2} \int_0^1 v \sin\left(\frac{u}{v}\right) \Big|_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_0^1 v \sin(1) - v \sin(-1)\, dv$$

$$= \sin(1) \int_0^1 v\, dv$$

$$= \frac{1}{2} \sin(1).$$

Question 2. [3 marks] Evaluate the line integral

$$\int_C z\,ds,$$

where C is the curve given by the parametric equations: $x = t\cos t$, $y = t\sin t$, $z = t$, $0 \le t \le 2\pi$.

Solution: $\frac{dx}{dt} = \cos t - t\sin t$, $\frac{dy}{dt} = \sin t + t\cos t$, $\frac{dz}{dt} = 1$.

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1^2}$$

$$= \sqrt{\cos^2 t \cancel{- 2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t \cancel{+ 2t\sin t\cos t} + t^2\cos^2 t + 1}$$

$$= \sqrt{1 + t^2 + 1}$$

$$= \sqrt{2 + t^2}$$

$$\int_C z\,ds = \int_0^{2\pi} t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} t\sqrt{2 + t^2}\,dt$$

$$= \frac{2}{3}\cdot\frac{1}{2}\,(2 + t^2)^{\frac{3}{2}}\Big|_0^{2\pi}$$

$$= \frac{1}{3}\left[(2 + 4\pi^2)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right].$$

Question 3. For all three parts of this question, $\vec{F}$ is the vector field defined by

$$\vec{F}(x,y,z) = (x^3 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z - xy)\vec{k}.$$

(a) [3 marks] Show that the vector field $\vec{F}$ is conservative by finding a potential function $f(x,y,z)$ such that $\vec{F} = \nabla f$.

Solution: $\vec{F} = \nabla f \iff \begin{cases} \frac{\partial f}{\partial x} = x^3 - yz \\ \frac{\partial f}{\partial y} = y^2 - xz \\ \frac{\partial f}{\partial z} = z - xy \end{cases}$

$$\frac{\partial f}{\partial x} = x^3 - yz \Rightarrow f(x,y,z) = \int \frac{\partial f}{\partial x} dx = \int x^3 - yz\, dx = \frac{x^4}{4} - xyz + C(y,z)$$

i.e. $f(x,y,z) = \frac{x^4}{4} - xyz + C(y,z)$.

$$\Rightarrow \underset{\substack{\| \\ y^2 - xz}}{\frac{\partial f}{\partial y}} = -xz + \frac{\partial C}{\partial y}(y,z) \Rightarrow \frac{\partial C}{\partial y}(y,z) = y^2 \Rightarrow C(y,z) = \frac{y^3}{3} + h(z)$$

i.e. $f(x,y,z) = \frac{x^4}{4} - xyz + \frac{y^3}{3} + h(z)$

$$\Rightarrow \underset{\substack{\| \\ z - xy}}{\frac{\partial f}{\partial z}} = -xy + h'(z) \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + C.$$

$$\Rightarrow f(x,y,z) = \frac{x^4}{4} - xyz + \frac{y^3}{3} + \frac{z^2}{2} + C.$$

Thus, $f(x,y,z) = \frac{x^4}{4} + \frac{y^3}{3} + \frac{z^2}{2} - xyz$ is a potential function of $\vec{F}$.

(b) [1 mark] Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is a smooth curve from the point $(0,0,0)$ to the point $(\sqrt{2}, \sqrt{3}, \sqrt{2})$.

By FTLI, $\int_C \vec{F} \cdot d\vec{r} = f(\sqrt{2}, \sqrt{3}, \sqrt{2}) - f(0,0,0)$

$$= \frac{(\sqrt{2})^4}{4} + \frac{(\sqrt{3})^3}{3} + \frac{(\sqrt{2})^2}{2} - \sqrt{2}\cdot\sqrt{3}\cdot\sqrt{2} - 0$$

$$= 1 + \sqrt{3} + 1 - 2\sqrt{3}$$

$$= 2 - \sqrt{3}$$

(c) [1 mark] Compute curl$\vec{F}$ and div$\vec{F}$.

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 - yz & y^2 - xz & z - xy \end{vmatrix}$$

$$= \vec{i}\left(-x - (-x)\right) - \vec{j}\left(-y - (-y)\right) + \vec{k}\left(-z - (-z)\right)$$

$$= \vec{0}.$$

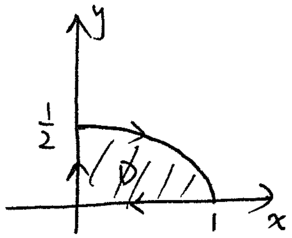
$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(y^2 - xz) + \frac{\partial}{\partial z}(z - xy)$$

$$= 3x^2 + 2y + 1.$$

Question 4. [4 marks] Use Green's theorem to evaluate

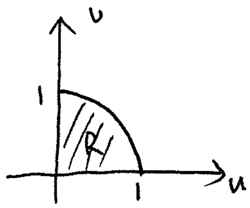
$$\int_C (x^2y + e^x)dx - (4xy^2 + \sin y)dy,$$

where C is the piecewise smooth curve enclosing the quarter of the ellipse $x^2 + 4y^2 \le 1$ lying in the first quadrant and traversed in the **clockwise** direction.



$$D = \{(x,y) \mid x^2+4y^2 \le 1, \quad x \ge 0,\ y \ge 0\}$$

$$T \quad \begin{cases} u = x \\ v = 2y \end{cases} \Leftrightarrow \begin{cases} x = u \\ y = \frac{1}{2}v \end{cases}$$



$$\int_C (x^2y + e^x)dx - (4xy^2 + \sin(y))dy$$

$$= -\iint_D \frac{\partial}{\partial x}\left(-(4xy^2 + \sin y)\right) - \frac{\partial}{\partial y}(x^2y + e^x)\, dA$$

$$= -\iint_D -4y^2 - x^2\, dA$$

$$= \iint_R (u^2+v^2)\frac{1}{2}\, dA$$

polar coordinates

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 r^3\,dr$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{4}$$

$$= \frac{\pi}{16}.$$