MATH 2X03, Sample Exam

Dr. Changliang Wang

Solutions

Date: June, 2018.

Duration of Exam: 2.5 hours

Family Name: ______________ **First Name:** ______________

Student Number: ______________________________

This examination paper contains 6 questions (with parts) on a total of 12 pages. You are responsible for ensuring that your copy of the exam paper is complete. Please bring any discrepancies to the attention of your invigilator.

INSTRUCTIONS:

1. Answer all questions in the spaces provided, and make sure that you justify clearly all steps in your solutions.
2. The only calculator that is permitted in this examination is the Casio FX 991MS or MS Plus calculator.

Question	Max Marks	Marks
1	6	
2	9	
3	8	
4	9	
5	7	
6	11	
Total	50	

Question 1. [6 marks] Decide whether the statements in parts (a) to (c) are true or false and circle your choices. **No justification needed.**

(a) [2 marks] Let $\vec{F}$ be a vector field of the form $\vec{F}(x,y,z) = (f(y,z), g(x,z), h(x,y))$. Then $\text{div}\vec{F} = 0$.

TRUE **FALSE**

(b) [2 marks] Let $\vec{F}$ be the vector field given by $\vec{F}(x,y,z) = (-y, x, z)$. There exists a function f such that $\vec{F} = \nabla f$.

TRUE **FALSE**

(c) [2 marks] If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^2$ whose component functions have continuous derivatives, then $\vec{F}$ is conservative if and only if $\int_C \vec{F} d\vec{r}$ is independent of path.

TRUE **FALSE**

Question 2. For both parts of this question, let S be the surface with parametric equation

$$\vec{r}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, \theta), 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}.$$

(a) [4 marks] Find an equation of the tangent plane of the surface S at the point $\vec{r}(\frac{1}{2}, \frac{\pi}{4})$.

Solution: $\vec{r}_\rho = (\cos\theta, \sin\theta, 0)$, $\vec{r}_\theta = (-\rho\sin\theta, \rho\cos\theta, 1)$.

$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -\rho\sin\theta & \rho\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, \rho)$$

$$(\vec{r}_\rho \times \vec{r}_\theta)(\tfrac{1}{2}, \tfrac{\pi}{4}) = (\sin(\tfrac{\pi}{4}), -\cos(\tfrac{\pi}{4}), \tfrac{1}{2}) = (\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{1}{2})$$

$$\vec{r}(\tfrac{1}{2}, \tfrac{\pi}{4}) = (\tfrac{1}{2}\cdot\cos(\tfrac{\pi}{4}), \tfrac{1}{2}\sin(\tfrac{\pi}{4}), \tfrac{\pi}{4}) = (\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4}, \tfrac{\pi}{4})$$

Thus, $\frac{\sqrt{2}}{2}(x - \frac{\sqrt{2}}{4}) - \frac{\sqrt{2}}{2}(y - \frac{\sqrt{2}}{4}) + \frac{1}{2}(z - \frac{\pi}{4}) = 0$ is an equation of the tangent plane.

Question 2 (cont'd)

(b) [5 marks] Evaluate the integral

$$\iint_S x dS.$$

<u>Solution</u>: By (a), $\vec{r}_\rho \times \vec{r}_\theta = (\sin\theta, -\cos\theta, \rho)$

So $|\vec{r}_\rho \times \vec{r}_\theta| = \sqrt{\sin^2\theta + (-\cos\theta)^2 + \rho^2} = \sqrt{1+\rho^2}$

Let $D = \{(\rho, \theta) \mid 0 \leq \rho \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}\}$

Then, $\displaystyle\iint_S x dS = \iint_D \rho\cos\theta \cdot |\vec{r}_\rho \times \vec{r}_\theta| \, dA$

$$= \iint_D \rho\cos\theta \cdot \sqrt{1+\rho^2} \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \cos\theta \, \rho\sqrt{1+\rho^2} \, d\rho \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \int_0^1 \rho\sqrt{1+\rho^2} \, d\rho$$

$$= \sin\theta \Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{2} \cdot \frac{2}{3} (1+\rho^2)^{\frac{3}{2}} \Big|_0^1$$

$$= 1 \cdot \frac{1}{3} \cdot \left[2^{\frac{3}{2}} - 1\right]$$

$$= \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right).$$

Question 3. [8 marks] Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = (z, zx, xy)$ and S is the surface given by $z = x \sin y$, $0 \le x \le 2$, $0 \le y \le \pi$, oriented downward.

Solution: Choose a parametrization of S as

$$\vec{r}(x,y) = (x, y, x\sin y), \qquad (x,y) \in D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le \pi\}$$

$$\vec{r}_x = (1, 0, \sin y), \qquad \vec{r}_y = (0, 1, x\cos y)$$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \sin y \\ 0 & 1 & x\cos y \end{vmatrix} = (-\sin y, -x\cos y, 1) \quad \leftarrow \text{upward normal.}$$

Because S is oriented downward:

$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_D \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dA$$

$$= -\iint_D (x\sin y, x^2 \sin y, xy) \cdot (-\sin y, -x\cos y, 1)\, dA$$

$$= -\iint_D -x\sin^2 y - x^3 \sin y \cos y + xy \, dA$$

$$= -\int_0^\pi \int_0^2 -x\sin^2 y - x^3 \sin y \cos y + xy \, dx\, dy$$

$$= -\int_0^\pi \left(-\frac{x^2}{2}\sin^2 y - \frac{x^4}{4}\sin y \cos y + \frac{x^2 y}{2} \right) \Bigg|_{x=0}^{x=2} dy$$

$$= -\int_0^\pi -2\sin^2 y - 4\sin y \cos y + 2y \, dy$$

$$= +\int_0^\pi 2\sin^2 y + 4 \sin y \cos y - 2y \, dy$$

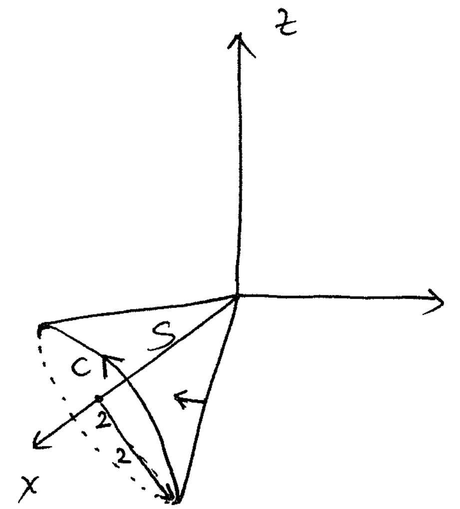
$$= \int_0^\pi 1 - \cos(2y) + 4\sin y \cos y - 2y \, dy$$

$$= y - \frac{1}{2}\sin(2y) + 2\sin^2 y - y^2 \Big|_0^\pi$$

$$= \pi - \pi^2$$

Question 4. [9 marks] Use Stokes' Theorem to evaluate $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = (\tan^{-1}(x^2yz^2), x^2y, x^2z^2)$, and S is the cone $x = \sqrt{y^2+z^2}$, $0 \le x \le 2$, oriented in the direction of the positive x-axis.

Solution:



The boundary of the surface is the circle C with radius 2 in the plane $x=2$. The orientation of C induced by the orientation of S is counterclockwise when viewed in the negative x-axis direction. (Indicated in the above picture).

Then we choose a parametrization for C as:

$\vec{r}(\theta) = (2, 2\cos\theta, 2\sin\theta)$, $0 \le \theta \le 2\pi$.

$\Rightarrow \vec{r}'(\theta) = (0, -2\sin\theta, 2\cos\theta)$.

→ This gives the orientation for C as in the picture.

Question 4 (cont'd)

By Stokes' Thm,

$$\iint_S \text{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(\theta))\cdot \vec{r}'(\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(\tan^{-1}(2^2\cdot 2\cos\theta\cdot 4\sin^2\theta),\ 4\cdot 2\cos\theta,\ 4\cdot 4\sin^2\theta\right)\cdot(0, -2\sin\theta, 2\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta\sin\theta + 32\sin^2\theta\cos\theta\, d\theta$$

$$= -8\sin^2\theta + \frac{32}{3}\sin^3\theta \Big|_0^{2\pi}$$

$$= 0$$

Question 5. [7 marks] Let $\vec{F}$ be the vector field given by

$$\vec{F}(x,y,z) = (y, xz, xy).$$

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by the parametric equations: $x = \cos t, y = \sin t, z = \tan t, 0 \le t \le \frac{\pi}{4}$.

Solution: $\vec{r}(t) = (\cos t, \sin t, \tan t), \quad 0 \le t \le \frac{\pi}{4}$

$\vec{r}'(t) = \left(-\sin t, \cos t, \frac{1}{\cos^2 t}\right)$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{4}} \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\, dt$$

$$= \int_0^{\frac{\pi}{4}} (\sin t, \sin t, \cos t \sin t)\cdot\left(-\sin t, \cos t, \frac{1}{\cos^2 t}\right) dt$$

$$= \int_0^{\frac{\pi}{4}} -\sin^2 t + \sin t \cos t + \frac{\sin t}{\cos t}\, dt$$

$$= \int_0^{\frac{\pi}{4}} -\frac{1-\cos(2t)}{2} + \sin t \cos t + \frac{\sin t}{\cos t}\, dt$$

$$= \int_0^{\frac{\pi}{4}} -\frac{1}{2} + \frac{1}{2}\cos(2t) + \sin t \cos t + \frac{\sin t}{\cos t}\, dt$$

$$= -\frac{1}{2}t + \frac{1}{4}\sin(2t) + \frac{1}{2}\sin^2 t - \ln|\cos t| \Big|_{t=0}^{t=\frac{\pi}{4}}$$

$$= -\frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{4}\sin\left(2\cdot\frac{\pi}{4}\right) + \frac{1}{2}\left(\sin\left(\frac{\pi}{4}\right)\right)^2 - \ln\left|\cos\left(\frac{\pi}{4}\right)\right| - (0+0+0-\ln 1)$$

$$= -\frac{\pi}{8} + \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} - \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\pi}{8} + \frac{1}{4} + \frac{1}{4} - \ln\left(\frac{\sqrt{2}}{2}\right)$$

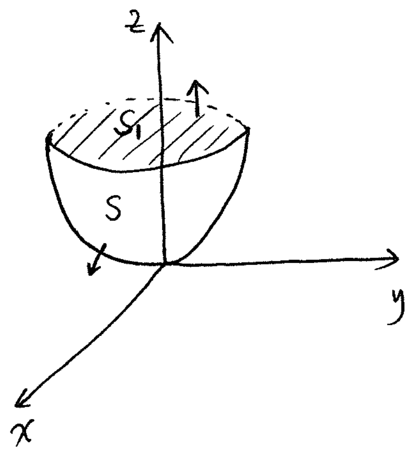
$$= -\frac{\pi}{8} + \frac{1}{2} - \ln\left(\frac{\sqrt{2}}{2}\right).$$

Question 6. [11 marks] Use the Divergence Theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}$ is the vector field given by

$$\vec{F}(x, y, z) = (\cos(y) + xy^2, x\sin(z), y + x^2 z),$$

and S is the part of the paraboloid $z = x^2 + y^2$ between the planes $z = 0$ and $z = 4$, oriented downward. (Note that S does not contain the interior of the circle $x^2 + y^2 = 4$ in the plane $z = 4$.)

Solution:



Let S_1 be the disc $x^2+y^2 \le 4$ in the plane $z=4$ with upward orientation.

Let E be the solid region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$. Then the boundary of E consists of S_1 and S with the given orientations.

upward downward.

By divergence Theorem:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\, dV.$$

$$\operatorname{div}\vec{F} = y^2 + 0 + x^2 = x^2+y^2.$$

$$\| $$

$$\frac{\partial}{\partial x}(\cos y + xy^2) + \frac{\partial}{\partial y}(x\sin z) + \frac{\partial}{\partial z}(y + x^2 z).$$

Question 6 (cont'd)

$$E=\{(x,y,z)\mid x^2+y^2\le z\le 4,\ 0\le x^2+y^2\le 4\}=\{(x,y,z)\mid x^2+y^2\le z\le 4,\quad (x,y)\in D\}$$

$$D=\{(x,y)\mid 0\le x^2+y^2\le 4\}.$$

$$\iiint_E \operatorname{div}\vec{F}\,dV=\iint_D\int_{x^2+y^2}^{4}(x^2+y^2)\,dz\,dA=\iint_D (x^2+y^2)z\Big|_{z=x^2+y^2}^{z=4}dA$$

$$=\iint_D 4(x^2+y^2)-(x^2+y^2)^2\,dA$$

$$=\int_0^{2\pi}\int_0^2(4r^2-r^4)\cdot r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^2 4r^3-r^5\,dr$$

$$=2\pi\left(r^4-\frac{r^6}{6}\right)\Big|_0^2$$

$$=2\pi\left(2^4-\frac{2^6}{6}\right)=\frac{32}{3}\pi.$$

Choose a parametrization of S_1 as: $\vec{r}(x,y)=(x,y,4)$, $(x,y)\in D$.

$\vec{r}_x=(1,0,0)$, $\vec{r}_y=(0,1,0)$ $\Rightarrow$ $\vec{r}_x\times\vec{r}_y=(0,0,1)\leftarrow$ upward normal

Thus, $$\iint_{S_1}\vec{F}\cdot d\vec{S}=\iint_D\vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x\times\vec{r}_y)\,dA=\iint_D y+x^2\cdot 4\,dA$$

$$=\int_0^{2\pi}\int_0^2(r\sin\theta+4r^2\cos^2\theta)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 r^2\sin\theta+4r^3\cos^2\theta\,dr\,d\theta$$

$$=\int_0^{2\pi}\frac{r^3}{3}\sin\theta+r^4\cos^2\theta\Big|_0^2\,d\theta$$

$$=\int_0^{2\pi}\frac{8}{3}\sin\theta+16\cos^2\theta\,d\theta$$

$$=\int_0^{2\pi}\frac{8}{3}\sin\theta+8+8\cos(2\theta)\,d\theta$$

$$=-\frac{8}{3}\cos\theta+8\theta+4\sin(2\theta)\Big|_0^{2\pi}$$

$$=16\pi.$$

Thus, $$\iint_S\vec{F}\cdot d\vec{S}=\iiint_E(\operatorname{div}\vec{F})\,dV-\iint_{S_1}\vec{F}\cdot d\vec{S}$$

$$=\frac{32\pi}{3}-16\pi$$

$$=-\frac{16}{3}\pi.$$

ROUGH WORK

THE END