Syzygies of Random Monomial Ideals

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Random Monomial Ideals

Definition

Define a *Random Stanley-Reisner Ideal* by taking the Stanley-Reisner ideal associated with a random flag complex, denoted by $\Delta(n, p)$.

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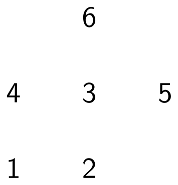

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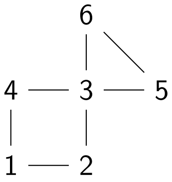

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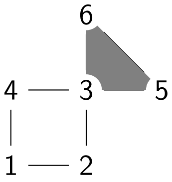

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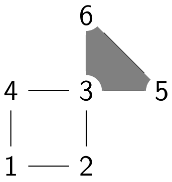

$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$

Syzygies

Definition

Let k be a field and M a module, then the betti numbers are

$$\beta_{i,j}(M,k) = \dim Tor_i(M,k)_j$$

These are then usually displayed as a table where the p-th column and q-th row is the $\beta_{p,p+q}$ entry

$$\begin{array}{|cccc}
\hline
\beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\
\beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

Theorem (Ein-Lazarsfeld, 2012)

Let X be a smooth, d-dimensional projective variety and let A be a very ample divisor on X. For any $n \geq 1$, let S_n be the homogeneous coordinate ring of X embedded by nA. For each $1 \leq k \leq d$, $\rho_k(S_n) \to 1$ as $n \to \infty$.

Definition

Let M be a module over a polynomial ring, then we define a "density" for the k-th row of the betti table as follows

$$\rho_k(M) := \frac{\#\{i \in [0, \mathrm{pdim}(M)] \text{ where } \beta_{i,i+k}(M) \neq 0\}}{\mathrm{pdim}(M) + 1}.$$

Asymptotic Syzygies

Question

What other families have the property that $\rho_k(I_n) \to 1$

- Intergral Varieties (Zhou '14)
- Arithmetically CM (Ein Erman Lazarsfeld '16)
- Iterated subdivision of Stanley Resiner rings (Conca Juhnke-Kubitzke Welker '14)

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- Random Stanley Reisner Ideals

Nonvanishing

Theorem (Erman-Y., 2017)

Fix i, j with $1 \leq i$ and $i+1 \leq j \leq 2i$ and let $r := j - i - 1$. Fix some constant $0 < \epsilon \leq \frac{1}{2}$ and let $\Delta \sim \Delta(n, p)$.

1. *If $\frac{1}{n^{1/r}} \ll p \leq \epsilon$ then $\mathbb{P}[\beta_{i,j}(S/I_\Delta) \neq 0] \to 1$.*
2. *If $p \ll \frac{1}{n^{1/r}}$ then $\mathbb{P}[\beta_{i,j}(S/I_\Delta) = 0] \to 1$.*

Corollary (Erman-Y., 2017)

Fix some $r \geq 1$. Let $\Delta \sim \Delta(n, p)$ with $\frac{1}{n^{1/r}} \ll p \ll 1$. For each $1 \leq k \leq r + 1$, we have

$$\rho_k(S/I_\Delta) \to 1$$

in probability.

Nonvanishing

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1. *If $\frac{1}{n^{1/r}} \ll p \leq \epsilon$ then $\mathbb{P}[\beta_{i,v}(S/I_\Delta) \neq 0] \to 1$.*
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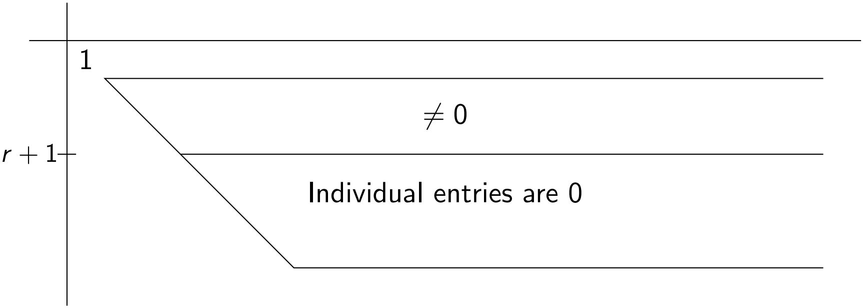


Sketch of Proof

For the first statement it suffices to show that $\mathbb{E}[\beta_{i,j}]$ goes to infinity. For that we'll apply Hochster's formula.

Theorem (Hochster's Formula)

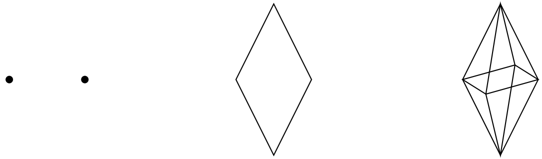
For Δ a simplicial complex and I_Δ the associated Stanley-Reisner Ideal

$$\beta_{i,j}(S/I_\Delta) = \sum_{|\alpha|=j} \dim \widetilde{H}_{i-j-1}(\Delta|_\alpha)$$

Lower Bound

We'll restrict specifically to the case where $j = 2i$ and $i = r + 1$ To find an lower bound on $\mathbb{E}[\beta_{i,j}]$, we consider the following particular flag complex

$$\diamondsuit_r = \text{r-fold suspension of 2-points}$$



$r = 0$ $\quad$ $r = 1$ $\quad$ $r = 2$

Lower Bound

Lemma

$\widetilde{H}_r(\diamondsuit_r) = \mathbb{Z}$ and $\diamondsuit_r$ has the fewest verticies of any flag complex with r-th homology.

Then

$$\begin{aligned}\mathbb{E}[\beta_{i,j}(S/I_\Delta)] &\geq \mathbb{E}[\#\text{ of }\diamondsuit_r\text{ as subcomplexes}]\\ &= Cn^{2r+2}p^{r(2r+2)}(1-p)^{r+1}\\ &= C(np^r)^{2r+2}(1-p)^{r+1}\end{aligned}$$

Then if $\frac{1}{n^{1/r}} \ll p \ll 1$, we get that

$$\mathbb{E}[\beta_{i,j}(S/I_\Delta)] \to \infty$$

Normality

There is another asymptotic behavior that we observed, in this case inspired by a result of Ein, Erman, and Lazarsfeld.

Theorem (Ein-Erman-Lazarsfeld 2012)

For C a smooth projective curve, with $C \xrightarrow{|A_n|} \mathbb{P}^N$. where the degrees of the A_n are increasing, then the first row of the betti table of converges to a binomial.

Normality

Theorem (Erman-Y., 2017)

Fix a constant $0 < c < 1$ *and let* $\Delta \sim \Delta(n, \frac{c}{n})$ *be a random flag complex. If* $\{i_n\}$ *is an integer sequence satisfying* $i_n = n/2 + o(n)$, *and if* $C := \frac{1-c}{2}$, *then*

$$\frac{\beta_{i_n, i_n+1}(S/I_\Delta)}{Cn\binom{n}{i_n}} \longrightarrow 1 \qquad \text{in probability.}$$

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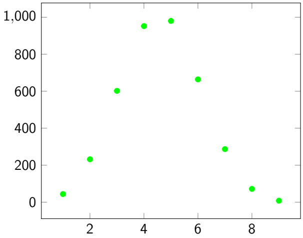

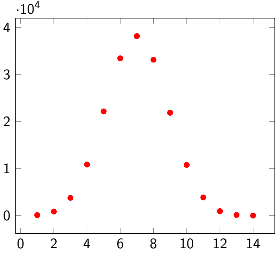

Sketch of Proof

Again I'll provide a lower bound, the uppper bound proceeds similarly.
Because we took $p = c/n$ for $0 < c < 1$ we can assume that our graph is always a collection of unicyclic components.

$$\begin{aligned}
\beta_{i,i+1}(S/I_\Delta) &= \sum_{|\alpha|=i} \widetilde{H}_0(\Delta|_\alpha) \\
&= \sum_{|\alpha|=i} i - E(\Delta|_\alpha) + C(\Delta|_\alpha) \\
&\geq \binom{n}{i} \cdot i - \sum_{|\alpha|=i} E(\Delta|_\alpha) \\
&= \binom{n}{i} \cdot i - \binom{n}{i-2} E(\Delta) \\
&\sim \binom{n}{i} (Cn)
\end{aligned}$$

Projective Dimension/CM

Theorem (Erman-Y., 2017)

For any $k \geq 1$, and any p satisfying $\frac{1}{n^{2/3}} \ll p \ll \left(\frac{\log(n)}{n}\right)^{2/(k+3)}$ we have that $\frac{\operatorname{codim}(S/I_\Delta)}{\operatorname{pdim}(S/I_\Delta)} \to 1$ in probability, yet the probability that S/I_Δ is Cohen-Macaulay goes to 0.

Other Applications of Randomness

- Random Toric Surfaces (Y. 2016)
- Other models of Random Monomial Ideals
  (De Loera,Petrovíc,Silverstein,Stasi, Wilburne 2017)

Thank You