Random Toric Surfaces and a Threshold for Smoothness

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Abstract

We present a notion of a *random toric surface* modeled on a notion of a random graph. We then study some threshold phenomena related to the smoothness of the resulting surfaces.

Motivation: Random Graphs

Definiton 1 (Erdős - Rényi Random Graph)**.** *The distribution $G(n,p)$ for a random graph is given by taking n verticies and for each potential edge, adding it with probability p.*

Theorem 2 (Erdős - Rényi 1959)**.** *For any $\epsilon > 0$ and for a graph G chosen with respect to $G(n,p)$*

1. *if $p \geq \frac{(1+\epsilon)\log n}{n}$ Then with high probability G is connected*
2. *if $p \leq \frac{(1-\epsilon)\log n}{n}$ Then with high probability G is disconnected*

Random Toric Surfaces

Definiton 3 (Random Fans)**.** *To define a distribution $T(h,p)$ start with the set of rays where the minimal lattice generator (a,b) has the property that $|a|, |b| \leq h$. Choose rays among these with probability p. Finally take the fan created by completing the set of chosen rays to a fan.*

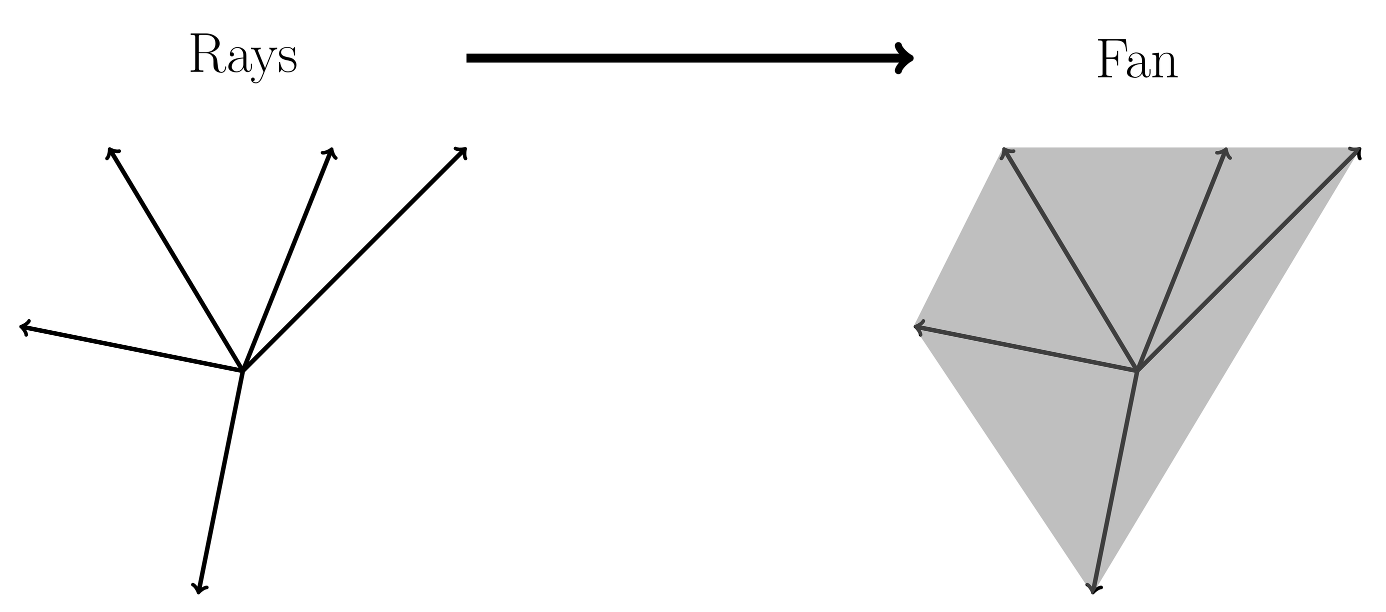

Figure 1: An example of the the process of completing rays to a fan by filling in cones

The Main Result

Theorem 4. *Let p be a function of h. For a fan Σ chosen with respect to $T(h,p)$, we have the following behavior*

1. *if $p \prec 1/h^2$ or $1-p \prec 1/h^2$ then with high probability $X(\Sigma)$ is smooth*
2. *if $p \succ 1/h^2$ and $1-p \succ 1/h^2$ then with high probability $X(\Sigma)$ is singular*

- The smooth case proceeds by noting that at either end of the spectrum, we get a particular smooth fan with high probability.
- For the singular case
  - A cone corresponds to a singular point if and only if the fundamental parallelogram has an interior lattice point.
  - The singular points on a random toric surface arise in two different ways

The Dense Singular Case

If the number of rays in a random fan grows faster than linearly in h, we can find cones in our fan which comes from blowing down along a single ray. This corresponds to certain configurations of rays illustrated below.

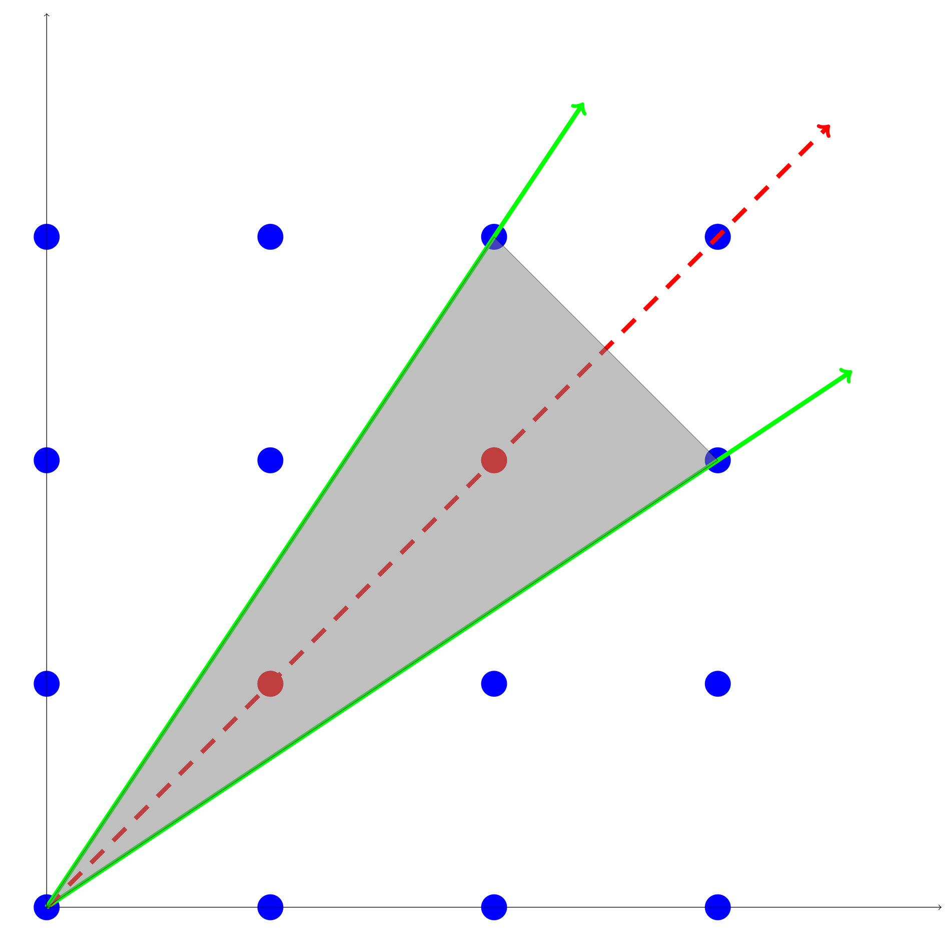

Figure 2: The result of removing the ray (1,1) from a fan with the lattice points giving the singularity highlighted in red

The Sparse Singular Case

If instead the number of rays grows slower than quadratically in h then we can exploit the relative lack of rays to show that there is always some ray where none of the potential neighboring rays which can make it smooth exist.

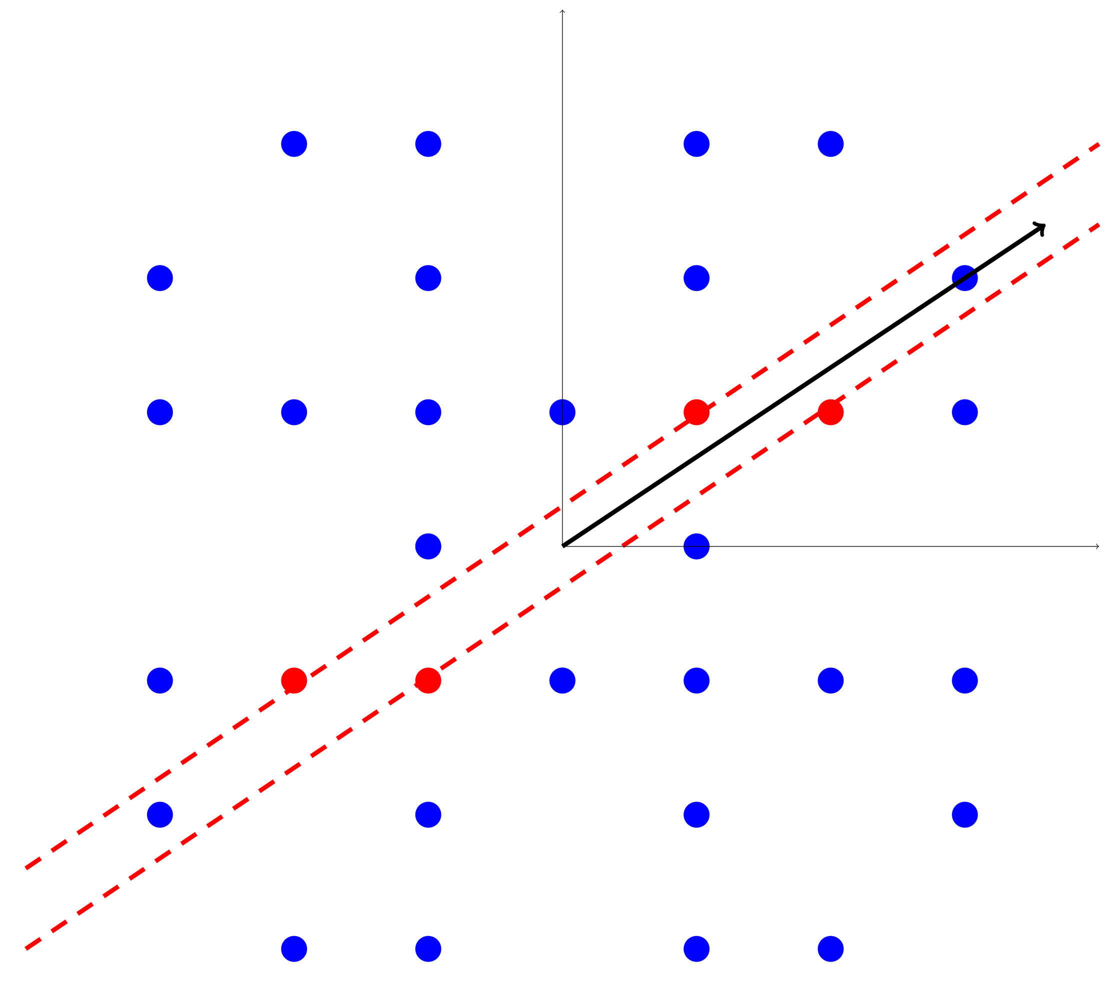

Figure 3: An illustration of the set of rays ρ for which $cone(\rho,(3,2))$ is smooth (in red) as compared to the set of all rays (in blue) with $|\rho| \leq 3$

Further Directions

- How do we generalize this to higher dimensions?
- What are other structures where random techniques might be fruitful?

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