Asymptotic Syzygies via Numerical Linear Algebra and High Throughput Computing

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Syzygies of the Veronese

We will consider the syzygies of the image of map

$$\mathbb{P}^2 \xrightarrow{|\mathcal{O}(d)|} \mathbb{P}^{\binom{d+2}{2}-1}$$

Which we will denote $K(0; d)$. As an extension we will also consider syzygies coming from the pushforward of linebundles $\mathcal{O}(b)$, which we will denote $K(b; d)$.

Goals

- Compute the multigraded entries of $K(b; d)$ for all b and as many d as possible using numerical techniques and high throughput computing.
- Develop techniques for using numerical linear algebra and distributed computation to compute betti numbers.
- Create an online resource containing all of the computed data.
- Investigate the Schur decomposition of $K(b; d)$.
- Investigate new conjectures about the Veronese.

Computing Syzygies

By noticing that $K_{p,q}(M) = \mathrm{Tor}(M,k)_{p,p+q}$ this gives two techniques

- Resolving M: We can compute a minimal free resolution of M, i.e.
$$M \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_n$$
  such that $\partial F_i \subseteq mF_{i-1}$
- Resolve k: Starting with the Koszul complex
$$k \leftarrow \bigwedge^1 R \leftarrow \bigwedge^2 R \leftarrow .. \leftarrow \bigwedge^n R.$$
  Then $\mathrm{Tor}(M,k)$ is computed by tensoring this complex with M and computing homology.

Computing Syzygies Using the Koszul Complex

Example

For $K_{9,0}(3;6)$ we need to compute the rank of the linear map

$$\bigwedge^9 S_6 \otimes S_3 \to \bigwedge^8 S_6 \otimes S_9$$

As it turns out this map is block diagonal with one block for every multidegree, so we can decompose it into 178 matrices taking a total of 2GB of space.

This is not a new technique, both Castryck, Cools, Demeyer, and Lemmens in 2016[3] and Greco and Martino in 2016[2], used this to compute syzygies.

Example

$K(0;5)$

1	—	···	—	—	—	—	—	—	—
—	165	···	134640	39780	?	?	—	—	—
—	—	···	—	?	?	2160	595	90	6

The Computation

Example

$K(0;5) \rightsquigarrow K(2;5)$

6	90	595	2160	4200	2002	—	⋯	—	—
—	—	—	375	4858	39780	134640	⋯	165	—
—	—	—	—	—	—	—	⋯	—	1

The Computation

Example

$K(0;5) \rightsquigarrow K(2;5)$

6	90	595	2160	4200	2002	–	⋯	–	–
–	–	–	375	4858	39780	134640	⋯	165	–
–	–	–	–	–	–	–	⋯	–	1

- Reduce via duality and Hilbert functions
- Compute matrices using code in C++
- Use existing sparse matrix code (LUSOL[1]) to compute the LU decomposition, which we use to compute the rank
- Use HTCondor, to run all of matrices involved in a distributed manner

Some Statistics

- For $d = 6$, generating the matrices requires $\sim$ 1 hour
- Largest completed matrices are $\sim$ 600,000x1,000,000
- They are sparse, with the matrices for $K_{p,q}(b;d)$ having p non-zero entries per row

Largest Computations

	max memory	max runtime	total runtime
$K_{5,0}(2;4)$	800MB	30 sec	10 min
$K_{8,0}(3;5)$	800MB	2 min	1 hr
$K_{8,0}(3;6)$	500GB	5 days	13 days

Post Processing

Note

The numerical linear algebra algorithms we use can have errors.

- Given the multigraded data we construct the Schur decomposition of $K_{p,q}$ greedily.
- This can detect and correct small errors.
- The data is then placed on our website `http://syzygydata.com`.

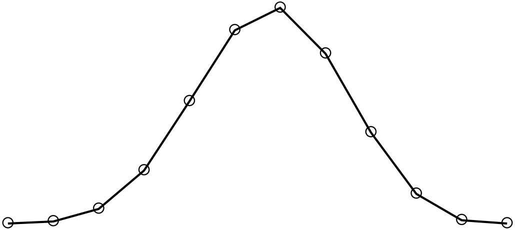
Conjectures

- We give a conjectural description of the most dominant weights of the Schur modules in $K_{p,q}(b; d)$
- We also provide evidence for a conjecture for the distribution of the values of the strands of $K(b; d)$

Normal Distribution Conjecture

Conjecture (Ein-Erman-Lazarsfeld)

On $\mathbb{P}^n$ fix q and b, then consider the function $p \mapsto \dim K_{p,q}(b;d)$. Then as $d \to \infty$, then with appropriate normalization, this function converges to a normal distribution.

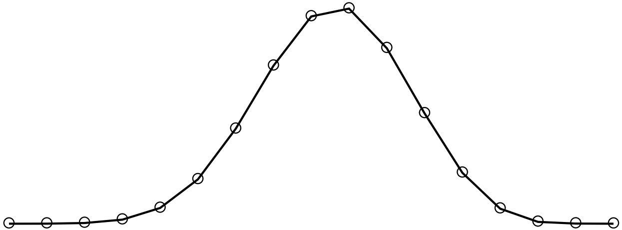


$p \mapsto \dim K_{p,1}(0;4)$

Normal Distribution Conjecture

Conjecture (Ein-Erman-Lazarsfeld)

On $\mathbb{P}^n$ *fix* q *and* b*, then consider the function* $p \mapsto \dim K_{p,q}(b; d)$*. Then as* $d \to \infty$*, then with appropriate normalization, this function converges to a normal distribution.*



$p \mapsto \dim K_{p,1}(0; 5)$

Normal Distribution Conjecture

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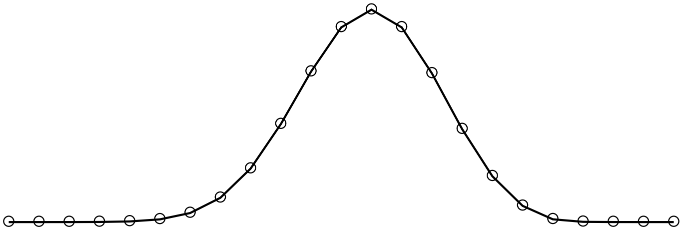

$p \mapsto \dim K_{p,1}(0; 6)$

Normal Distribution Conjecture

Conjecture (Ein-Erman-Lazarsfeld)

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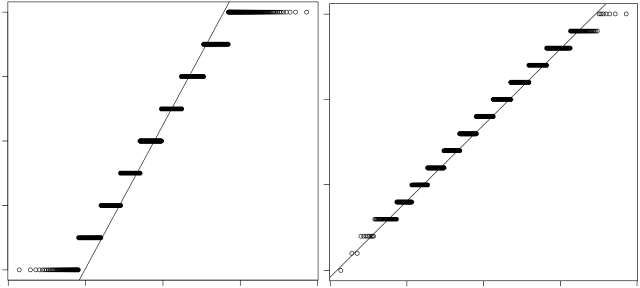

References

[1] Michael Saunders, *LUSOL*, https://web.stanford.edu/group/SOL/software/lusol/.

[2] Ornella Greco and Ivan Martino, *Syzygies of the Veronese modules*, Comm. Algebra **44** (2016), no. 9, 3890–3906.

[3] Wouter Castryck, Filip Cools, Jeroen Demeyer, and Alexander Lemmens, *Computing graded Betti tables of toric surfaces*, ArXiv (2016), available at arXiv:1606.08181.