Complex Analysis -Assignment Four

due Tuesday, Feb27

1. Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Hint: Use the rectangle contour introduced in class.

2. Suppose that u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \left(\frac{\pi}{\sin(\pi u)}\right)^2.$$

Hint: Integrating

$$f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2}$$

over the circle $|z| = N + \frac{1}{2}$ for integer $N \geq |u|$, adding the residues of f inside the circle, and letting N tend to infinity.

3. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes no where else in $|z| \leq 1$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that if ε is sufficiently small, then $f_\varepsilon(z)$ has a unique zero in $|z| < 1$.

4. (Bonus) Prove the identity

$$(*) \qquad \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{-2\pi |x|} dx = \frac{1}{\pi} \frac{1}{1 + |\xi|^2}.$$

Step 1. Show that

$$e^{-\alpha} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{1+x^2} dx, \qquad \alpha > 0. \tag{0. 1}$$

Look at your Assignment Two, question 1, and deduce the result.

Step2. Use (0. 1) and

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u} du$$

together with the fact that

$$\int_{-\infty}^\infty e^{-2\pi \mathbf{i} x\xi} e^{-\pi x^2} dx = e^{-\pi\xi^2} \tag{0. 2}$$

to show

$$e^{-\alpha} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\alpha^2/4u} du. \tag{0. 3}$$

Step3. Use both (0. 2) and (0. 3) to derive the identity (∗) and finish the problem.

Have a nice break!