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Classical solutions to differential inclusions with totally disconnected right-hand side

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Abstract

Let $F = F(t,x)$ be a bounded, Hausdorff continuous multifunction with compact, totally disconnected values. Given any $y_0 \in F(t_0, x_0)$, we show that the differential inclusion $\dot{x} \in F(t,x) \subset \mathbb{R}^m$ has a globally defined classical solution, with $x(t_0) = x_0$, $\dot{x}(t_0) = y_0$.

1. Introduction

Let $F = F(t,x)$ be a bounded, Hausdorff continuous multifunction defined on $\mathbb{R} \times \mathbb{R}^m$, whose values are non-empty compact subsets of $\mathbb{R}^m$. In this paper we study the existence of classical (i.e. continuously differentiable) solutions to the Cauchy problem

$$\dot{x} \in F(t,x), \quad t \in [0,T], \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where the upper dot denotes a derivative w.r.t. time.

If F is convex-valued, then there exists a continuous selection $f(t,x) \in F(t,x)$. By Peano's theorem, the ODE $\dot{x} = f(t,x)$ admits at least one solution taking the initial value (1.2). In turn, this provides a classical solution to (1.1). Even when the sets $F(t,x)$ are not convex, it is known that the Cauchy problem (1.1)–(1.2) always admits a solution in the Carathéodory sense, i.e. an

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absolutely continuous function $x(\cdot)$ that satisfies (1.1) at almost every time $t \in [0, T]$. This result was first proved in [8] by a clever choice of polygonal approximations, then in [1] by means of continuous selections from a multivalued Picard map in $\mathbf{L}^1$. Alternative proofs rely on directionally continuous selections [3], or on a set-valued version of the Baire category theorem [5] where points are replaced by compact sets.

In general, however, the lack of convexity prevents the existence of classical solutions. A well-known example is provided by the multifunction $G : [0, 1] \mapsto \mathbb{R}^2$, defined as

$$G(t) = \begin{cases} \{(\cos\theta, \sin\theta);\ \theta \in [0, 2\pi]\} & \text{if } t = 0, \\ \{(\cos\theta, \sin\theta);\ \theta \in [t^{-1}, t^{-1} + 2\pi - t]\} & \text{if } 0 < t \leqslant 1. \end{cases}$$

In this case, G is continuous w.r.t. the Hausdorff metric, with compact but non-convex values. Since G does not admit any continuous selection, the differential inclusion $\dot{x} \in G(t)$ cannot have any continuously differentiable solution defined on the entire interval [0, 1].

In the classic paper [9] it was proved that the differential inclusion (1.1) admits $\mathcal{C}^1$ solutions provided that the multifunction F is Lipschitz continuous w.r.t. the Hausdorff metric. We recall that, even in the Lipschitz continuous case, a continuous selection may not exist. For example, the multifunction $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ described by

$$F(r\cos\theta, r\sin\theta) = \left\{ \left(r\cos\frac{\theta}{2}, r\sin\frac{\theta}{2} \right), \left(-r\cos\frac{\theta}{2}, -r\sin\frac{\theta}{2} \right) \right\}, \quad r \geqslant 0,\ \theta \in [0, 2\pi],$$

is Lipschitz continuous but it does not admit a continuous selection on any neighborhood of the origin.

Aim of the present note is to prove the existence of uniformly continuous selections for a continuous multifunction $t \mapsto G(t)$, and of classical solutions for the differential inclusion (1.1), relying on an assumption which is quite the opposite of convexity. Namely, we shall assume that our multifunctions have totally disconnected values. We recall that a set $K \subset \mathbb{R}^m$ is totally disconnected if, for every $a, b \in K$ with $a \neq b$ one can find disjoint open sets A, B such that $a \in A$, $b \in B$ and $K \subseteq A \cup B$. For example, every countable subset $K \subset \mathbb{R}^m$ is totally disconnected. Cantor-like or fractal sets provide many other examples of uncountable, totally disconnected sets. Our main results are as follows.

Theorem 1. *Let $G : [0, T] \mapsto \mathbb{R}^m$ be a Hausdorff continuous multifunction. Assume that each set $G(t)$ is compact and totally disconnected. Then the family of continuous selections $t \mapsto g(t) \in G(t)$ is a non-empty, compact, totally disconnected subset of $\mathcal{C}^0([0, T]; \mathbb{R}^m)$. Indeed, for every $y_0 \in G(0)$ there exists a continuous selection such that $x(0) = y_0$.*

Theorem 2. *Let $F : [0, T] \times \mathbb{R}^m \mapsto \mathbb{R}^m$ be a bounded, Hausdorff continuous multifunction with compact, totally disconnected values. Then for every $x_0 \in \mathbb{R}^m$ and $y_0 \in F(0, x_0)$ there exists a $\mathcal{C}^1$ solution $t \mapsto x(t)$ of the Cauchy problem* (1.1)–(1.2)*, such that $\dot{x}(0) = y_0$. The family of all such classical solutions is a compact subset of $\mathcal{C}^1([0, T]; \mathbb{R}^m)$.*

The proofs of these theorems will be given in Sections 2 and 3, respectively. Finally, in Section 4 we discuss the connectedness of the set of classical solutions.

A special case of the above theorems, valid for multifunctions with finitely many values and additional structure, was recently proved in [10,11]. See also [7] for a related result. For the basic properties of set-valued functions and differential inclusions we refer to [2]. The book [12] contains an extensive collection of results on continuous selections. An interesting survey of the theory of differential inclusions can be found in [6].

2. Proof of Theorem 1

Given $y_0 \in G(0)$, a continuous selection $t \mapsto g(t) \in G(t)$ will be obtained as limit of a sequence of polygonal approximations $g_n(\cdot)$. The proof will be worked out in several steps.

1. For each $n \geqslant 1$, let $\delta_n \doteq T/n$. To construct the piecewise affine approximation g_n, we first define its values at the points $t_{n,j} \doteq j\delta_n = jT/n$, by induction on $j = 0, 1, \ldots, nT$.

(i) We begin by defining $g_n(0) = g_n(t_{n,0}) = y_0$.

(ii) Next, assume that the values $g_n(t_{n,j}) \in G(t_{n,j})$ have been defined for all $j = 0, 1, \ldots, k$. We then select a value $g_n(t_{n,k+1}) \in G(t_{n,k+1})$ as close as possible to $g_n(t_{n,k})$. In other words,

$$\big|g_n(t_{n,k+1}) - g_n(t_{n,k})\big| = \min_{y \in G(t_{n,k+1})} \big|y - g_n(t_{n,k})\big|. \tag{2.1}$$

Notice that the minimum in (2.1) is certainly attained, because the set $G(t_{n,j+1})$ is compact.

By induction, the values $g_n(t_{n,j})$ are thus defined for all $j = 0, 1, \ldots, nT$. We then extend the function g_n to the entire interval $[0, T]$ by setting

$$g_n(t) \doteq g_n(t_{n,k-1}) + \frac{g_n(t_{n,k}) - g_n(t_{n,k-1})}{t_{n,k} - t_{n,k-1}} \cdot t, \qquad t \in [t_{n,k-1}, t_{n,k}]. \tag{2.2}$$

2. We claim that the distance of the graphs of g_n from the graph of G approaches zero as $n \to \infty$. More precisely, call

$$\operatorname{Graph} G \doteq \big\{(t, y);\ t \in [0, T],\ y \in G(t)\big\}.$$

Then

$$\lim_{n \to \infty} \sup_{t \in [0,T]} d\big(\big(t, g_n(t)\big); \operatorname{Graph} G\big) = 0. \tag{2.3}$$

Indeed, since G is a continuous multifunction defined on a compact interval, it is uniformly continuous w.r.t. the Hausdorff metric. Therefore, there exists a continuous, increasing function $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$\omega(0) = 0, \qquad d_H\big(G(t), G(s)\big) \leqslant \omega\big(|t - s|\big). \tag{2.4}$$

Now consider any $t \in [0, T]$ and $n \geqslant 1$. To fix the ideas, assume $t \in [t_{n,k}, t_{n,k+1}]$. By our construction it follows $(t_k, g_n(t_{n,k})) \in \operatorname{Graph} G$. Therefore, (2.1)–(2.2) yield

$$\begin{aligned} d\big(\big(t, g_n(t)\big); \operatorname{Graph} G\big) &\leqslant d\big(\big(t, g_n(t)\big); \big(t_{n,k}, g_n(t_{n,k})\big)\big) \\ &\leqslant |t - t_{n,k}| + \big|g_n(t) - g_n(t_k)\big| \\ &\leqslant |t - t_{n,k}| + \big|g_n(t_{k+1}) - g_n(t_k)\big| \end{aligned}$$

$$\begin{aligned} &= |t - t_{n,k}| + d\big(g_n(t_k), G(t_{n,k+1})\big) \\ &\leqslant \delta_n + \omega(\delta_n). \end{aligned} \tag{2.5}$$

Since $\delta_n \to 0$, it is clear that the right-hand side of (2.5) approaches zero as $n \to \infty$, uniformly w.r.t. $t \in [0, T]$. This establishes (2.4).

3. We claim that our approximate selections $g_n(\cdot)$ are equicontinuous: for any given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t - s| \leqslant \delta \quad \Longrightarrow \quad \big|g_n(t) - g_n(s)\big| \leqslant \varepsilon \quad \text{for all } t, s \in [0, T],\ n \geqslant 1. \tag{2.6}$$

The claim will be proved by contradiction. If the polygonal functions $g_n(\cdot)$ were not equicontinuous, we could find $\varepsilon > 0$ and sequences of times $s_k, t_k \in [0, T]$ and indices $n(k)$, $k = 1, 2, \ldots,$ such that

$$\big|g_{n(k)}(t_k) - g_{n(k)}(s_k)\big| \geqslant \varepsilon, \qquad |t_k - s_k| \leqslant \frac{1}{k} \quad \text{for all } k \geqslant 1. \tag{2.7}$$

By possibly taking a subsequence, we can assume $s_k \to \tau$, $t_k \to \tau$, for some point $\tau \in [0, T]$. We can also assume that $n(k) \to \infty$ as $k \to \infty$. Indeed, if $n(k) \leqslant M$ for all k, then the inequalities in (2.7) would imply that the set of finitely many continuous functions $\{g_1, \ldots, g_M\}$ is not equicontinuous, i.e. a contradiction.

Because of (2.3), there exists points $a, b \in G(\tau)$ such that, by possibly extracting a further subsequence, we have the convergence

$$g_{n(k)}(s_k) \to a, \qquad g_{n(k)}(t_k) \to b \quad \text{as } k \to \infty. \tag{2.8}$$

Notice that (2.7) implies $|b - a| \geqslant \varepsilon$, hence $a \neq b$.

We now recall that the set $G(\tau)$ is compact and totally disconnected. Therefore, there exists disjoint open sets A, B such that

$$a \in A,\ b \in B, \quad G(\tau) \subset A \cup B.$$

Since $G(\tau)$ is compact, by possibly shrinking A and B we can also assume that these two sets are strictly separated:

$$\sigma \doteq \inf\big\{|x - y|;\ x \in A,\ y \in B\big\} > 0. \tag{2.9}$$

Since the multifunction G is Hausdorff continuous, there exists $\rho_0 > 0$ such that

$$G(t) \subset A \cup B \quad \text{for all } t \in [\tau - 2\rho_0, \tau + 2\rho_0] \cap [0, T]. \tag{2.10}$$

For each $k \geqslant 1$, consider the scalar function

$$\phi_k(t) \doteq d\big(g_{n(k)}(t); A\big) - d\big(g_{n(k)}(t); B\big).$$

Recalling (2.8), since A and B are open, for all k sufficiently large we have

$$t_k, s_k \in [\tau - \rho_0, \tau + \rho_0], \qquad g_{n(k)}(s_k) \in A, \qquad g_{n(k)}(t_k) \in B. \tag{2.11}$$

Together, (2.11) and (2.9) imply

$$\phi_k(s_k) \leqslant -\sigma, \qquad \phi_k(t_k) \geqslant \sigma.$$

By the intermediate value theorem, there exists $\eta_k \in [s_k, t_k]$ such that $\phi_k(\eta_k) = 0$. This implies

$$\min_{t\in[\tau-2\rho_0,\tau+2\rho_0]} d\big(g_{n(k)}(\eta_k), G(t)\big) \geqslant \min\big\{d\big(g_{n(k)}(\eta_k), A\big), d\big(g_{n(k)}(\eta_k), B\big)\big\} \geqslant \frac{\sigma}{2}.$$

If now $(t, y) \in \operatorname{Graph} G$, we consider two possibilities:
Case 1: $t \in [\tau - 2\rho_0, \tau + 2\rho_0]$. Then by (2.10) it follows

$$\big|g_{n(k)}(\eta_k) - y\big| \geqslant d\big(g_{n(k)}(\eta_k); A \cup B\big) \geqslant \frac{\sigma}{2}.$$

Case 2: $t \notin [\tau - 2\rho_0, \tau + 2\rho_0]$. Then by (2.11) we have $|\eta_k - t| \geqslant \rho_0$.

In both cases, for all k sufficiently large we thus have

$$d\big(\big(\eta_k, g_{n(k)}(\eta_k)\big); \operatorname{Graph} G\big) \geqslant \min\left\{\frac{\sigma}{2}, \rho_0\right\}. \tag{2.12}$$

Since the right-hand side of (2.12) is a positive constant independent of k, we obtain a contradiction with (2.3). This proves the uniform continuity of the sequence of approximate selections $g_n(\cdot)$.

4. By the Ascoli–Arzelà compactness theorem, the sequence $g_n(\cdot)$ admits a subsequence which converges to a continuous function $g(\cdot)$ uniformly on $[0, T]$. It remains to prove that g is a selection, i.e. $g(t) \in G(t)$ for all $t \in [0, T]$. This is clear, because G has closed graph and, for each $t \in [0, T]$,

$$d\big(\big(t, g(t)\big), \operatorname{Graph} G\big) = \lim_{n\to\infty} d\big(\big(t, g_n(t)\big), \operatorname{Graph} G\big) = 0,$$

because of (2.3).

5. Finally, we show that the family $\mathcal{S}$ of all continuous selections is a compact, totally disconnected subset of $\mathcal{C}^0([0, T]; \mathbb{R}^m)$. The compactness of the sets $G(t)$ implies that $\mathcal{S}$ is closed, while the boundedness of G implies that all selections $g(\cdot)$ are uniformly bounded. Moreover, the same argument used in step **3** now shows that the selections $g \in \mathcal{S}$ are equicontinuous. Hence, by the Ascoli–Arzelà theorem, $\mathcal{S}$ is compact in the $\mathcal{C}^0$ topology.

To prove that $\mathcal{S}$ is totally disconnected, let $g_1 \neq g_2$ be two distinct selections. Then $g_1(\tau) \neq g_2(\tau)$ for some $\tau \in [0, T]$. Since $G(\tau)$ is totally disconnected, we can find disjoint open sets A, B such that

$$G(\tau) \subset A \cup B, \qquad g_1(\tau) \in A, \qquad g_2(\tau) \in B.$$

We can now write $\mathcal{S} \doteq \mathcal{S}_1 \cup \mathcal{S}_2$ with

$$S_1 \doteq \big\{g \in \mathcal{S};\ g(\tau) \in A\big\}, \qquad S_2 \doteq \big\{g \in \mathcal{S};\ g(\tau) \in B\big\}.$$

Clearly, $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint sets, relatively open in the $\mathcal{C}^0$ topology. This shows that $\mathcal{S}$ is totally disconnected, completing the proof.

3. Proof of Theorem 2

Let the initial point $x_0 \in \mathbb{R}^m$ and the initial velocity $y_0 \in F(0, x_0)$ be given. The existence of a $\mathcal{C}^1$ solution to (1.1) with the required initial data will be proved in several steps.

1. For $n \geqslant 1$, set $\delta_n \doteq T/n$. We begin by constructing a sequence of approximate solutions $x_n \in \mathcal{C}^1([0, T]; \mathbb{R}^m)$ with the following properties:

$$x_n(0) = x_0, \qquad \dot{x}_n(t) = y_0, \quad t \in [0, \delta_n], \tag{3.1}$$

$$\dot{x}_n(t) \in F\big(t - \delta_n, x_n(t - \delta_n)\big), \quad t \in [\delta_n, T]. \tag{3.2}$$

For each fixed $n \geqslant 1$, the function $x_n(\cdot)$ is constructed as follows. On the first interval $[0, \delta_n]$ we define

$$x_n(t) = x_0 + y_0 t, \quad t \in [0, \delta_n]. \tag{3.3}$$

Assume now that $x_n(\cdot)$ has been defined, is continuously differentiable, and satisfies (3.1)–(3.2) on the interval $[0, k\delta_n]$, for some $k \in \{1, \ldots, n-1\}$. In order to extend this function to the larger interval $[0, (k+1)\delta_n]$, we construct a continuous selection

$$t \mapsto y_n(t) \in F\big((t - \delta_n), x_n(t - \delta_n)\big), \quad t \in \big[k\delta_n, (k+1)\delta_n\big], \tag{3.4}$$

with initial value $y_n(k\delta_n) = \dot{x}_n(k\delta_n-)$. Note that this is possible because of Theorem 1. Then we define

$$x_n(t) = x_n(k\delta_n) + \int_{k\delta_n}^{t} y_n(\tau) \cdot d\tau. \tag{3.5}$$

By induction on $k = 1, 2, \ldots, n-1$, the function $x_n(\cdot)$ can thus be constructed on the entire interval $[0, T]$. Since the multifunction F is uniformly bounded, the maps $x_n(\cdot)$ are uniformly Lipschitz continuous. Observing that

$$\big(t - \delta_n, x_n(t - \delta_n), \dot{x}_n(t)\big) \in \operatorname{Graph} F \quad \text{for all } t \in [\delta_n, T],$$

we conclude that

$$\sup_{t \in [0,T]} \operatorname{dist}\big(\big(t, x_n(t), \dot{x}_n(t)\big); \operatorname{Graph} F\big) = \mathcal{O}(1) \cdot \delta_n. \tag{3.6}$$

In particular, this maximum distance approaches zero as $n \to \infty$.

2. Next, assume that a sequence of approximations $x_n(\cdot)$ has been constructed. Since these maps are uniformly bounded and Lipschitz continuous, we can apply the Ascoli–Arzelà compactness theorem. By possibly taking a subsequence we can thus assume that the maps $x_n(\cdot)$ converge to some limit trajectory $x^*(\cdot)$ uniformly on $[0, T]$.

3. Next, we claim that the sequence of derivatives $y_n(t) = \dot{x}_n(t)$ is equicontinuous. This is proved by contradiction, by the same arguments used in the proof of Theorem 1. Assume that equicontinuity fails. Then we can find $\varepsilon > 0$ and sequences of times $s_k, t_k \in [0, T]$ and indices $n(k)$, $k = 1, 2, \ldots$, such that

$$\left|y_{n(k)}(t_k) - y_{n(k)}(s_k)\right| \geqslant \varepsilon, \qquad |t_k - s_k| \leqslant \frac{1}{k} \quad \text{for all } k \geqslant 1. \tag{3.7}$$

By possibly taking a subsequence, we can assume $s_k \to \tau$, $t_k \to \tau$ for some point $\tau \in [0, T]$. We can also assume that $n(k) \to \infty$ as $k \to \infty$. Indeed, if $n(k) \leqslant M$ for all k, then the inequalities in (3.7) would imply that the set of finitely many continuous functions $\{y_1, \ldots, y_M\}$ is not equicontinuous, i.e. a contradiction.

By possibly extracting a further subsequence, as $k \to \infty$ we can assume the convergence

$$s_n \to \tau, \qquad t_k \to \tau, \qquad y_{n(k)}(s_k) \to a, \qquad y_{n(k)}(t_k) \to b, \tag{3.8}$$

for some $\tau \in [0, T]$ and some values a, b such that $|b - a| \geqslant \varepsilon$. By the properties of the approximating sequence we now have

$$y_{n(k)}(s_k) \in F\big(s_k - \delta_{n(k)}, x_{n(k)}\big(s_k - \delta_{n(k)}\big)\big), \qquad y_{n(k)}(t_k) \in F\big(t_k - \delta_{n(k)}, x_{n(k)}\big(t_k - \delta_{n(k)}\big)\big).$$

Moreover, we have the convergence

$$x_{n(k)}(s_k) \to x^*(\tau), \qquad x_{n(k)}(t_k) \to x^*(\tau).$$

By the continuity of the multifunction F it thus follows that

$$a, b \in F\big(\tau, x^*(\tau)\big). \tag{3.9}$$

We now recall that the set $F(\tau, x^*(\tau))$ is totally disconnected. Therefore, there exists disjoint open sets A, B such that

$$a \in A, \ b \in B, \quad F\big(\tau, x^*(\tau)\big) \subset A \cup B.$$

Since $F(\tau, x^*(\tau))$ is compact, by possibly shrinking A and B we can also assume that these two sets are strictly separated:

$$\sigma \doteq \inf\big\{|x - y|; \ x \in A, \ y \in B\big\} > 0. \tag{3.10}$$

Since the multifunction F is Hausdorff continuous, there exists $r_0 > 0$ such that

$$F(t, x) \subset A \cup B \quad \text{for all } t \in [\tau - 2r_0, \tau + 2r_0] \cap [0, T], \ x \in B\big(x^*(\tau), 2r_0\big). \tag{3.11}$$

Furthermore, since the maps $x_n(\cdot)$ are uniformly Lipschitz continuous and $x_n(\tau) \to x^*(\tau)$, we can find $\rho_0 > 0$ such that

$$x_{n(k)}(t) \in B\big(x^*(\tau), r_0\big) \quad \text{for all } t \in [\tau - \rho_0, \tau + \rho_0] \cap [0, T], \tag{3.12}$$

provided that k is large enough. As in the proof of Theorem 1, consider the scalar functions

$$\phi_k(t) \doteq d\big(y_{n(k)}(t); A\big) - d\big(y_{n(k)}(t); B\big).$$

By the intermediate value theorem, there exists $\eta_k \in [s_k, t_k]$ such that $\phi_k(\eta_k) = 0$. This implies

$$d\big(y_{n(k)}(\eta_k), A \cup B\big) = \min\big\{d\big(y_{n(k)}(\eta_k), A\big), d\big(y_{n(k)}(\eta_k), B\big)\big\} \geqslant \frac{\sigma}{2}. \tag{3.13}$$

If now $(t, x, y) \in \operatorname{Graph} F$, we consider two possibilities:
Case 1: $|t - \tau| \leqslant 2r_0$ and $|x - x^*(\tau)| \leqslant 2r_0$. Then by (3.11) and (3.13) it follows

$$\big|y_{n(k)}(\eta_k) - y\big| \geqslant d\big(y_{n(k)}(\eta_k); A \cup B\big) \geqslant \frac{\sigma}{2}.$$

Case 2: Either $|t - \tau| > 2r_0$, or $|x - x^*(\tau)| > 2r_0$. Then by (3.12) and the convergence $\eta_k \to \tau$ we have either $|\eta_k - t| \geqslant \rho_0$, or $|x - x_{n(k)}(\eta_k)| \geqslant r_0$.

In both cases, for all k sufficiently large we thus have

$$d\big(\big(\eta_k, g_{n(k)}(\eta_k)\big), \operatorname{Graph} G\big) \geqslant \min\left\{\frac{\sigma}{2}, r_0, \rho_0\right\}. \tag{3.14}$$

Since the right-hand side of (3.14) is a positive constant independent of k, we obtain a contradiction with (3.6). This proves the equicontinuity of the sequence $y_n(\cdot)$.

4. Having proved that the time derivatives $y_n = \dot{x}_n$ are equicontinuous and uniformly bounded, by a compactness argument we can now extract a uniformly convergent subsequence, say $y_{n(k)} \to y^*$. Clearly, $y^*(t) = \dot{x}^*(t)$ for all $t \in [0, T]$. Moreover, by (3.3) we have $y_n(0) = y_0$ for all $n \geqslant 1$. Hence the initial condition $\dot{x}^*(0) = y^*(0) = y_0$ is satisfied.

Finally, recalling (3.6), we conclude that $(t, x^*(t), y^*(t)) \in \operatorname{Graph} F$ for all $t \in [0, T]$, because the graph of F is closed. Hence (1.1) holds. This concludes the proof of the first part of the theorem.

5. Finally, consider the set of all classical solutions $x(\cdot)$ of the Cauchy problem (1.1)–(1.2). By the previous arguments, the set of first derivatives $\dot{x}(\cdot)$ of these solutions is uniformly bounded and equicontinuous, hence relatively compact in $\mathcal{C}^0([0, T]; \mathbb{R}^m)$. Since the graph of F is closed, the second statement in Theorem 2 is now clear.

4. Topology of the solution set

Let $G : [0, T] \mapsto \mathbb{R}^m$ be a continuous multifunction with compact, totally disconnected values. Consider the Cauchy problem

$$x(0) = 0, \qquad \dot{x}(t) \in G(t), \quad t \in [0, T]. \tag{4.1}$$

Its classical solutions are precisely the mappings

$$t \mapsto x^g(t) = \int_0^t g(s)\,ds, \tag{4.2}$$

where $g(\cdot)$ is a continuous selection of the multifunction $G(\cdot)$.

By Theorem 1, the family $\mathcal{S}$ of all these continuous selections is a compact, totally disconnected subset of $\mathcal{C}^0([0,T];\mathbb{R}^m)$. Since the map $g(\cdot) \mapsto x^g(\cdot)$ defined at (4.2) is continuous and one-to-one, it is a homeomorphism from the compact set $\mathcal{S}$ onto its image. We thus conclude that the family of all classical solutions of (4.1) is a compact, totally disconnected subset of $\mathcal{C}^0([0,T];\mathbb{R}^m)$.

Next, let $F : R^m \mapsto \mathbb{R}^m$ be a bounded, continuous multifunction with compact, totally disconnected values. In this setting, it is natural to ask whether the set of all classical solutions to the Cauchy problem

$$x(0) = 0, \qquad \dot{x}(t) \in F\big(x(t)\big), \quad t \in [0,T], \tag{4.3}$$

is again totally disconnected.

If the multifunction F is only continuous, a counterexample is easy to find. Indeed, take $F(x) \doteq \{2\sqrt{|x|}\}$. In this case, the differential inclusion reduces to an ODE with continuous right-hand side,

$$x(0) = 0, \qquad \dot{x}(t) = 2\sqrt{|x|}, \quad t \in [0,T]. \tag{4.4}$$

It is well known that this Cauchy problem has a connected set of solutions.

In the following, we show that the solution set of (4.3) need not be totally disconnected, even if the multifunction F is Lipschitz continuous. The following result will be used.

Lemma 1. *Let $G : \mathbb{R}^m \mapsto \mathbb{R}^m$ be a Lipschitz continuous, compact valued multifunction, so that*

$$d_H\big(G(x), G(x')\big) \leqslant L|x - x'| \quad \textit{for all } x, x' \in \mathbb{R}^m. \tag{4.5}$$

Let $\varphi : \mathbb{R}^m \mapsto \mathbb{R}^m$ be a continuous map with Lipschitz constant λ, and define a second multifunction F by setting

$$y \in F(x) \quad \textit{if and only if} \quad y \in F\big(x - \varphi(y)\big).$$

If $L\lambda < 1$, then the multifunction F is Lipschitz continuous with constant $\widehat{L} = L/(1 - L\lambda)$.

Proof. Assume $y \in F(x)$, and let a second point x' be given. We claim that there exists $y' \in F(x')$ such that

$$|y - y'| \leqslant \frac{L}{1 - L\lambda}|x - x'|. \tag{4.6}$$

Indeed, by assumption $y \in G(x - \varphi(y))$. Since G is Lipschitz, we can find $y_1 \in G(x' - \varphi(y))$ such that $|y_1 - y| \leqslant L|x' - x|$.

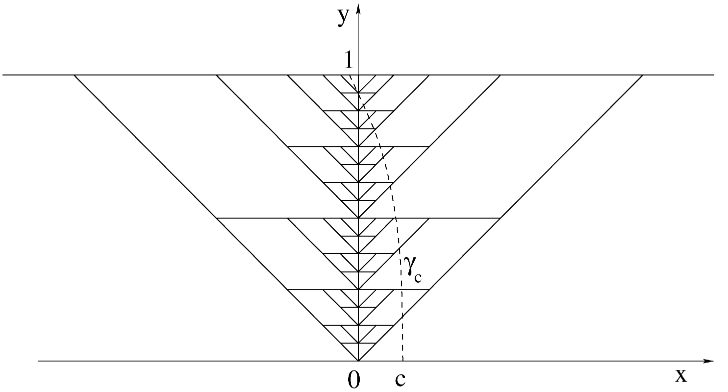

Fig. 1.

Next, since G and φ are Lipschitz, we can find $y_2 \in G(x' - \varphi(y_1))$ such that

$$|y_2 - y_1| \leqslant L\big|\varphi(y_1) - \varphi(y)\big| \leqslant L \cdot L\lambda|x' - x|.$$

By induction, we obtain a sequence $y_1, y_2, \ldots$ such that

$$y_{k+1} \in G\big(x' - \varphi(y_k)\big), \tag{4.7}$$

$$|y_{k+1} - y_k| \leqslant L\big|\varphi(y_k) - \varphi(y_{k-1})\big| \leqslant L\lambda|y_k - y_{k-1}| \leqslant L^{k+1}\lambda^k|x - x'|. \tag{4.8}$$

By the assumption $L\lambda < 1$, the sequence $(y_k)_{k\geqslant 1}$ is Cauchy. Hence it converges to some limit point y'. By (4.7) and the continuity of F, φ it follows

$$y' \in G\big(x' - \varphi(y')\big) = F(x').$$

Finally, (4.8) yields

$$|y - y'| \leqslant |y - y_1| + \sum_{k=1}^{\infty} |y_{k+1} - y_k| \leqslant L\big(1 + L\lambda + L^2\lambda^2 + L^3\lambda^3 + \cdots\big)|x - x'|,$$

proving (4.6). □

Example. We are now ready to construct a Lipschitz continuous multifunction $F : \mathbb{R} \mapsto \mathbb{R}$ with totally disconnected values, such that the set of all classical solutions of (4.3) is not totally disconnected. As a first step, consider the multifunction $G : \mathbb{R} \mapsto \mathbb{R}$ whose graph is the following closed set (the solid lines in Fig. 1):

$$\text{Graph}(G) = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2,$$

where

$$\mathcal{G}_0 \doteq \{(x, y);\ x \in \mathbb{R},\ y = 1\} \cup \{(x, y);\ x = 0,\ y \in [0, 1]\},$$
$$\mathcal{G}_1 \doteq \bigcup_{n \geqslant 1} \{(x, y);\ x \in [-2^{-n}, 2^{-n}],\ y = k \cdot 2^{-n},\ k = 1, 2, 3, \ldots, n\},$$
$$\mathcal{G}_2 \doteq \bigcup_{n \geqslant 1} \{(x, y);\ x \in [-2^{-n}, 2^{-n}],\ y = |x| + k \cdot 2^{-n},\ k = 1, 2, 3, \ldots, n-1\}.$$

It is clear that each $G(x)$ is a compact subset of [0, 1]. Moreover, the map $x \mapsto G(x)$ is Lipschitz continuous with Lipschitz constant $L = 1$.

We now consider the Lipschitz map

$$\varphi(y) \doteq \begin{cases} 0 & \text{if } y < 0, \\ y^2/4 & \text{if } y \in [0, 1], \\ 1/4 & \text{if } y > 1. \end{cases}$$

Since φ is Lipschitz continuous with constant $\lambda = 1/2$, by Lemma 1 there exists a unique multifunction $F : \mathbb{R} \mapsto \mathbb{R}$ with Lipschitz constant $\widehat{L} = 1/(1 - 1/2) = 2$ such that

$$y \in F(x) \quad \text{if and only if} \quad y \in G\bigl(x - \varphi(y)\bigr).$$

Observe that each parabola $\gamma_c = \{(x, y);\ x = c - y^2/4\}$ (the broken line in Fig. 1) intersects the graph of G at countably many points. Hence, for each $c \in \mathbb{R}$, the set $F(c)$ is countable, and thus totally disconnected. We now observe that, for $x \in [0, 1/4]$ the multifunction F admits the continuous selection $f(x) = 2\sqrt{x}$. Indeed,

$$2\sqrt{x} \in [0, 1] = G(0) = G\left(x - \frac{(2\sqrt{x})^2}{4}\right).$$

In particular, the family of all classical solutions to (4.3) contains a nontrivial connected set of solutions of the ODE (4.4). For this example, we thus conclude that the set of all classical solutions is neither connected, nor totally disconnected.

Remark. If instead of $\mathcal{C}^1$ solutions one considers all Carathéodory solutions to the Cauchy problem (1.1)–(1.2), then this larger set of solutions is always a connected subset of $\mathcal{C}^0([0, T]; \mathbb{R}^m)$. See [4] for a proof of this result, which is valid more generally for lower semicontinuous multifunctions $F(\cdot)$ with compact, non-convex values.

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