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ON THE STABILITY OF THE BEST REPLY MAP FOR NONCOOPERATIVE DIFFERENTIAL GAMES

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Consider a differential game for two players in infinite time horizon, with exponentially discounted costs. A pair of feedback controls $(u_1^*(x), u_2^*(x))$ is Nash equilibrium solution if u_1^* is the best strategy for Player 1 in reply to u_2^*, and u_2^* is the best strategy for Player 2, in reply to u_1^*. The aim of the present note is to investigate the stability of the best reply map: $(u_1, u_2) \mapsto (\mathcal{R}_1(u_2), \mathcal{R}_2(u_1))$. For linear-quadratic games, we derive a condition which yields asymptotic stability, within the class of feedbacks which are affine functions of the state $x \in \mathbb{R}^n$. An example shows that stability is lost, as soon as nonlinear perturbations are considered.

Keywords: Differential games; Nash equilibrium; optimal feedback; best reply map.

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1. Introduction

Consider noncooperative differential game for two-players, where the state of the system evolves according to

$$\dot{x} = f(x, u_1, u_2). \tag{1.1}$$

Here, $x \in \mathbb{R}^n$ while the control functions $u_1(\cdot)$ and $u_2(\cdot)$ range over the domains $u_1 \in U_1 \subseteq \mathbb{R}^{m_1}$, $u_2 \in U_2 \subseteq \mathbb{R}^{m_2}$. The upper dot denotes a derivative with respect to time. The goal of each player is to minimize his own cost functional

$$J_i = J_i(x, u) = \int_0^\infty e^{-\gamma t} L_i(x(t), u_1(t), u_2(t))\, dt, \quad i = 1, 2. \tag{1.2}$$

This corresponds to an infinite horizon problem, with exponentially discounted cost.

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Given a feedback strategy $x \mapsto u_1(x)$ implemented by Player 1, we say that the feedback strategy $x \mapsto u_2^\sharp(x)$ is a *best reply* for Player 2, and write $u_2^\sharp \in \mathcal{R}_2(u_1)$, if $u_2^\sharp(\cdot)$ is optimal for the problem

$$\text{minimize:} \quad J_2 = \int_0^\infty e^{-\gamma t} L_2(x(t), u_1^*(x(t)), u_2)\, dt, \tag{1.3}$$

with dynamics

$$\dot{x} = f(x, u_1(x), u_2). \tag{1.4}$$

More precisely, we are here requiring that

(i) for every initial state $x_0 \in \mathbb{R}^n$, the Cauchy problem

$$\dot{x} = f(x, u_1(x), u_2^\sharp(x)), \quad x(0) = x_0, \tag{1.5}$$

has at least one Carathéodory solution, defined for all times $t \geq 0$.

(ii) Every Carathéodory solution of (1.5) is optimal in the sense that, given the initial state x_0 and the feedback $u_1(\cdot)$, the control $u_2^\sharp(\cdot)$ minimizes the cost J_2 among all strategies $u_2(\cdot)$ available to Player 2.

Given a feedback strategy $x \mapsto u_2(x)$ implemented by Player 2, a best reply $u_1^\sharp \in \mathcal{R}_1(u_2)$, for Player 1 is defined in a similar way.

Two feedback strategies $u_1 = u_1^*(x)$, $u_2 = u_2^*(x)$ constitute a *Nash equilibrium solution* to the differential game (1.1)–(1.2) if at the same time one has

$$u_1^* \in \mathcal{R}_1(u_2^*), \quad u_2^* \in \mathcal{R}_2(u_1^*). \tag{1.6}$$

One can regard the Nash solution as a fixed point of the "best reply" map

$$(u_1, u_2) \;\mapsto\; (\mathcal{R}_1(u_2), \mathcal{R}_2(u_1)). \tag{1.7}$$

Assuming that this "best reply" is unique, a natural question is whether the fixed point is asymptotically stable. In other words, let $(u_1^{(0)}, u_2^{(0)})$ be a pair of feedback controls sufficiently close to a Nash equilibrium (u_1^*, u_2^*), and define the iterates

$$\big(u_1^{(k+1)}, u_2^{(k+1)}\big) \doteq \big(\mathcal{R}_1(u_2^{(k)}), \mathcal{R}_2(u_1^{(k)})\big).$$

Is it true that $(u_1^{(k)}, u_2^{(k)}) \;\to\; (u_1^*, u_2^*)$ as $k \to \infty$? In the present note we provide a positive answer for a class of games with linear dynamics and quadratic costs, as long as the feedback controls $u_i^{(k)}(\cdot)$ are affine functions of the state $x \in \mathbb{R}^n$. In general, this convergence is not expected to hold for nonlinear systems. Indeed, we show by an example that, even for a linear-quadratic game, the fixed point of the best reply map is not stable with respect to nonlinear perturbations of the feedback controls, even if these perturbations have uniformly bounded support and are arbitrarily small in any $\mathcal{C}^k$ norm.

The paper is organized as follows: Section 2 reviews the basic theory of linear-quadratic optimal control problems in infinite time horizon, focusing on how the optimal feedback changes as a consequence of small variations in the dynamics and in the cost function. In Sec. 3, we briefly review the equations determining a Nash equilibrium solution to a linear-quadratic differential game.

After these preliminaries, Sec. 4 analyzes the stability of the best reply maps for linear-quadratic games, within the class of feedback controls which are affine functions of the state. Finally, in Sec. 5, we provide an example showing that, even for a simple linear-quadratic game, the Nash equilibrium feedbacks are not stable with respect to nonlinear perturbations.

For a general introduction to differential games we refer to [1, 5]. Linear-quadratic games have an extensive literature; see for example [6, 11] and references therein. For nonlinear problems, the papers [7, 8] analyze the relations between solutions to differential games and optimal feedback controls. Examples of feedback solutions to nonlinear differential games in infinite time horizon were studied in [4, 9]. For a special class of "nearly decoupled" games, an iterative procedure yielding a Nash solution can be found in [10].

2. Review of the Linear-Quadratic Optimal Control Problem

Consider a system on $\mathbb{R}^n$, with dynamics

$$\dot{x} = Ax + Bu + \mathbf{c} \tag{2.1}$$

and a cost functional

$$J = J(x,u) \doteq \int_0^\infty e^{-\gamma t}\left[\frac{1}{2}x^\dagger Px + x^\dagger Qu + \frac{|u|^2}{2} + \mathbf{a}^\dagger x + \mathbf{b}^\dagger u\right]dt. \tag{2.2}$$

Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are column vectors, while the superscript † denotes transposition. Moreover,

$$A, P \in \mathbb{R}^{n\times n}, \quad B, Q \in \mathbb{R}^{n\times m}, \quad \mathbf{a}, \mathbf{c} \in \mathbb{R}^n, \quad \mathbf{b} \in \mathbb{R}^m.$$

Without loss of generality, we can assume $P = P^\dagger$. We also remark that, if $R \in \mathbb{R}^{m\times m}$ is a symmetric, strictly positive definite matrix, then the more general cost functional

$$J \doteq \int_0^\infty e^{-\gamma t}\left[\frac{1}{2}x^\dagger Px + x^\dagger Qu + \frac{1}{2}u^\dagger Ru + \mathbf{a}^\dagger x + \mathbf{b}^\dagger u\right] dt$$

can be reduced to (2.2). Indeed, it suffices to use a new set of control variables $v = \Lambda u$, with $\Lambda^\dagger\Lambda = R$. This yields

$$|v|^2 = u^\dagger\Lambda^\dagger\Lambda u = u^\dagger Ru.$$

Let V be the value function for the optimization problem (2.1)–(2.2). More precisely, let $t \mapsto x(t, u, \bar{x})$ be the solution to (2.1) corresponding to the control function $u(\cdot)$ and the initial data

$$x(0) = \bar{x}. \tag{2.3}$$

We then define

$$V(\bar{x}) \doteq \inf_{u(\cdot)} J(x(u,\bar{x}),\, u). \tag{2.4}$$

Calling $\xi^\dagger \doteq \nabla V(x) = (V_{x_1}, \ldots, V_{x_n})$ the row vector describing the gradient of V, the optimal feedback control u^* is provided by

$$u^*(x,\xi) = \operatorname*{argmin}_{\omega} \left\{ \xi^\dagger B\omega + x^\dagger Q\omega + \frac{|\omega|^2}{2} + \mathbf{b}^\dagger \omega \right\}. \tag{2.5}$$

Differentiating with respect to ω, we obtain

$$\xi^\dagger B + x^\dagger Q + u^{*\dagger} + \mathbf{b}^\dagger = 0, \quad u^*(x,\xi) = -(B^\dagger \xi + Q^\dagger x + \mathbf{b}). \tag{2.6}$$

It is well known that, if the value function V is a continuously differentiable function, then it provides a solution to the Hamilton–Jacobi equation

$$\gamma V = \nabla V \cdot \dot{x}^* + L(x, u^*), \tag{2.7}$$

where

$$\dot{x}^* = Ax + Bu^* + \mathbf{c}, \tag{2.8}$$

$$L(x,u^*) \doteq \frac{1}{2} x^\dagger P x + x^\dagger Q u^* + \frac{|u^*|^2}{2} + \mathbf{a}^\dagger x + \mathbf{b}^\dagger u^*. \tag{2.9}$$

After some calculations, this yields

$$\begin{aligned} \gamma V &= \nabla V \cdot [Ax - B(B^\dagger (\nabla V)^\dagger + Q^\dagger x + \mathbf{b}) + \mathbf{c}] \\ &\quad + \frac{1}{2} x^\dagger P x + \mathbf{a}^\dagger x - (x^\dagger Q + \mathbf{b}^\dagger)(B^\dagger (\nabla V)^\dagger + Q^\dagger x + \mathbf{b}) \\ &\quad + \frac{1}{2}((\nabla V)B + x^\dagger Q + \mathbf{b}^\dagger)(B^\dagger (\nabla V)^\dagger + Q^\dagger x + \mathbf{b}). \end{aligned} \tag{2.10}$$

This can be written as

$$\gamma V = \frac{1}{2} \nabla V \cdot \widetilde{A} \cdot (\nabla V)^\dagger + \nabla V \cdot \widetilde{B} x + \nabla V \cdot \tilde{\mathbf{c}} + \frac{1}{2} x^\dagger \widetilde{P} x + \tilde{\mathbf{q}}^\dagger x + \tilde{r}, \tag{2.11}$$

where

$$\begin{cases} \widetilde{A} = -BB^\dagger, & \widetilde{B} = A - BQ^\dagger, \quad \tilde{\mathbf{c}} = \mathbf{c} - B\mathbf{b}, \\ \widetilde{P} = P - QQ^\dagger, & \tilde{\mathbf{q}} = \mathbf{a} - Q\mathbf{b}, \quad \tilde{r} = -\dfrac{1}{2}\mathbf{b}^\dagger \mathbf{b}. \end{cases} \tag{2.12}$$

Now assume

$$V(x) = \frac{1}{2} x^\dagger M x + \mathbf{n}^\dagger x + e \tag{2.13}$$

for some symmetric matrix M, some vector $\mathbf{n}$ and a scalar e. This implies

$$\nabla V(x) = x^\dagger M + \mathbf{n}^\dagger. \tag{2.14}$$

Inserting (2.13)–(2.14) in (2.11), one obtains

$$\frac{\gamma}{2}x^\dagger Mx + \gamma\mathbf{n}^\dagger x + \gamma e = \frac{1}{2}(x^\dagger M + \mathbf{n}^\dagger)\widetilde{A}(Mx + \mathbf{n}) + (x^\dagger M + \mathbf{n}^\dagger)\widetilde{B}x + (x^\dagger M + \mathbf{n}^\dagger)\tilde{\mathbf{c}} + \frac{1}{2}x^\dagger \widetilde{P}x + \tilde{\mathbf{q}}^\dagger x + \tilde{r}. \tag{2.15}$$

Equating homogeneous terms of the same degree, and observing that $x^\dagger M\widetilde{B}x = x^\dagger \widetilde{B}^\dagger Mx$ because M is symmetric, we finally obtain

$$\begin{cases} \gamma M = M\widetilde{A}M + M\widetilde{B} + \widetilde{B}^T M + \widetilde{P}, \\ \gamma \mathbf{n}^\dagger = \mathbf{n}^\dagger \widetilde{A}M + \mathbf{n}^\dagger \widetilde{B} + \tilde{\mathbf{c}}^\dagger M + \tilde{\mathbf{q}}^\dagger, \\ \gamma e = \dfrac{|\mathbf{n}|^2}{2} + \mathbf{n}^\dagger\tilde{\mathbf{c}} + \tilde{r}. \end{cases} \tag{2.16}$$

The first equation in (2.16) is a quadratic equation for the $n \times n$ symmetric matrix M, whose solution depends on the matrices A, B, P, Q. Existence and uniqueness of solutions is not guaranteed, in general. As soon as the matrix M is determined, the last two equations in (2.16) yield the values of the vector $\mathbf{n}$ and of the scalar e. These last two equations are linear with respect to $\mathbf{n}, e$.

Next, assume that a solution of Eqs. (2.16) has been found. According to (2.6) the optimal feedback control is

$$u^*(x) = -[B^\dagger(\nabla V)^\dagger + Q^\dagger x + \mathbf{b}] = -[B^\dagger(Mx + \mathbf{n}) + Q^\dagger x + \mathbf{b}]. \tag{2.17}$$

The system dynamics is thus described by

$$\dot{x} = Ax - B[B^\dagger(Mx + \mathbf{n}) + Q^\dagger x + \mathbf{b}] + \mathbf{c}. \tag{2.18}$$

Our basic assumption will be that this dynamics is asymptotically stable. Equivalent conditions for this to happen are:

(i) The linear homogeneous system

$$\dot{x} = Yx, \quad Y \doteq A - BB^\dagger M - BQ^\dagger \tag{2.19}$$

is asymptotically stable.

(ii) All the eigenvalues of the matrix Y in (2.19) have strictly negative real part.

(iii) There exist constants $C, \omega > 0$ such that norm of the exponential matrix satisfies

$$\|e^{tY}\| \le Ce^{-t\omega} \quad \text{for all } t \ge 0. \tag{2.20}$$

We write (2.17) in the form

$$u^*(x) = Ux + \mathbf{v}, \tag{2.21}$$

where the $m \times n$-matrix U and the m-vector $\mathbf{v}$ are given by

$$U = -B^\dagger M + Q^\dagger, \quad \mathbf{v} = -B^\dagger\mathbf{n} + \mathbf{b}. \tag{2.22}$$

Clearly, this optimal feedback u^* depends on the coefficients $A, B, \mathbf{c}$ of the system (2.1) and on the coefficients $P, Q, \mathbf{a}, \mathbf{b}$ of the cost functional (2.2). Keeping $B, Q, \mathbf{b}$ constant, we wish to understand how $U, \mathbf{v}$ depend on the remaining parameters $A, P, \mathbf{c}, \mathbf{a}$. In other words, we study the differential of the map

$$(A, \mathbf{c}, P, \mathbf{a}) \overset{\Phi}{\mapsto} (U, \mathbf{v}). \tag{2.23}$$

Assume that we replace $(A, \mathbf{c}, P, \mathbf{a})$ by $(A + \varepsilon A', \mathbf{c} + \varepsilon \mathbf{c}', P + \varepsilon P', \mathbf{a} + \varepsilon \mathbf{a}')$. Then, the optimal feedback will have the form

$$u^*_\varepsilon(x) = (U + \varepsilon U')x + (\mathbf{v} + \varepsilon \mathbf{v}') + o(\varepsilon). \tag{2.24}$$

We seek an expression for $(U', \mathbf{v}')$ as a linear function of $A', P', \mathbf{c}', \mathbf{a}'$, depending on $A, P, \mathbf{c}, \mathbf{a}$. Inserting

$$M_\varepsilon = M + \varepsilon M' + o(\varepsilon)$$

inside the first equation in (2.16) and recalling (2.12), one obtains

$$\begin{aligned} \gamma M_\varepsilon = &-M_\varepsilon BB^\dagger M_\varepsilon + M_\varepsilon(A + \varepsilon A' - BQ^\dagger) + (A + \varepsilon A' - BQ^\dagger)^\dagger M_\varepsilon \\ &+ (P + \varepsilon P' - QQ^\dagger). \end{aligned}$$

Collecting terms of order ε we find

$$\begin{aligned} \gamma M' &= -MBB^\dagger M' - M'BB^\dagger M + M'(A - BQ^\dagger) \\ &\quad + (A - BQ^\dagger)^\dagger M' + MA' + A'M + P', \end{aligned}$$

$$\begin{aligned} \gamma M' &+ [M'(BB^\dagger M - A + BQ^\dagger)] + [M'(BB^\dagger M - A + BQ^\dagger)]^\dagger \\ &= (MA') + (MA')^\dagger + P'. \end{aligned}$$

We thus seek an $n \times n$ symmetric matrix M' such that

$$\gamma M' - M'Y - (M'Y)^\dagger = S, \tag{2.25}$$

where

$$Y \doteq A - BB^\dagger M - BQ^\dagger, \quad S \doteq (MA') + (MA')^\dagger + P'. \tag{2.26}$$

Notice that S is symmetric, but Y may not be symmetric. Of course, the left-hand side of (2.25) is always symmetric. The solution to (2.25) is provided by the formula

$$M' = \int_0^\infty e^{-\gamma t}(e^{tY})^\dagger S e^{tY}\, dt. \tag{2.27}$$

Notice that the integral is absolutely convergent, because of (2.20). Observing that M' is symmetric, the above formula can be verified by writing

$$\gamma M' - M'Y - (M'Y)^\dagger = -\int_0^\infty \frac{d}{dt}(e^{-\gamma t}(e^{tY})^\dagger S e^{tY})\, dt = S. \tag{2.28}$$

According to (2.22), the first order variations of the matrix U and of the vector $\mathbf{v}$ are computed by

$$U' = -B^\dagger M', \quad \mathbf{v}' = -B^\dagger \mathbf{n}'. \tag{2.29}$$

The symmetric matrix M' has already been computed at (2.27). Notice that M' depends only on the first order variations A', P' and not on $\mathbf{c}', \mathbf{a}'$.

For future use, we seek an expression for $\mathbf{n}'$ in the special case where $A' = 0$, $P' = 0$. By (2.12), the second equation in (2.16) yields

$$\mathbf{n} = (\gamma I + MBB^\dagger - A^\dagger + QB^\dagger)^{-1}(M(\mathbf{c} - B\mathbf{b}) + \mathbf{a} - Q\mathbf{b}).$$

Hence,

$$\mathbf{n}' = (\gamma I + MBB^\dagger - A^\dagger + QB^\dagger)^{-1}(M\mathbf{c}' + \mathbf{a}'). \tag{2.30}$$

Notice that the matrix

$$\gamma I + MBB^\dagger - A^\dagger + QB^\dagger = \gamma I - Y^\dagger$$

is certainly invertible. Indeed, the stability assumption (2.20) implies all of its eigenvalues have real part $> \gamma$.

3. A Linear-Quadratic Differential Game

We now consider a differential game for two players, with controls $u_1 \in \mathbb{R}^{m_1}$, $u_2 \in \mathbb{R}^{m_2}$. In place of (2.1), the dynamics is now described by

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + \mathbf{f}. \tag{3.1}$$

Assume that the cost functionals for the two players have the form

$$J_i = \int_0^\infty e^{-\gamma t} \left[\frac{1}{2} x^\dagger P_i x + \mathbf{a}_i^\dagger x + \frac{|u_i|^2}{2} + \sum_{j=1,2} (x^\dagger Q_{ij} u_j + \mathbf{b}_{ij}^\dagger u_j) \right] dt, \quad i = 1, 2. \tag{3.2}$$

Assume that the value functions V_1, V_2 are second order polynomials in the space variables $x_1, \ldots, x_n$:

$$V_i(x) = \frac{1}{2} x^\dagger M_i x + \mathbf{n}_i^\dagger x + e_i, \quad i = 1, 2, \tag{3.3}$$

so that

$$\nabla V_i(x) = (M_i x + \mathbf{n}_i)^\dagger. \tag{3.4}$$

Then, the optimal feedback controls u_1^*, u_2^* for the two players are determined by

$$\begin{aligned} u_i^*(x) &= \operatorname*{argmin}_{\omega}\left\{\nabla V_i \cdot B_i\omega + \frac{|\omega|^2}{2} + x^\dagger Q_{ii}\omega + \mathbf{b}_{ii}^\dagger\omega\right\} \\ &= -(B_i^\dagger(M_i x + \mathbf{n}_i) + Q_{ii}^\dagger x + \mathbf{b}_{ii}). \end{aligned} \tag{3.5}$$

The strategies (3.5) yield a Nash equilibrium solution in feedback form provided that the value functions V_1, V_2 satisfy the system of P.D.E's

$$\gamma V_i = \nabla V_i \cdot \dot{x} + L_i(x, u_1^*, u_2^*), \quad i = 1, 2, \tag{3.6}$$

where

$$\dot{x} = Ax + B_1 u_1^* + B_2 u_2^* + \mathbf{f}, \tag{3.7}$$

$$L_i(x, u_1, u_2) = \frac{1}{2}x^\dagger P_i x + \mathbf{a}_i^\dagger x + \frac{|u_i|^2}{2} + \sum_{j=1,2}(x^\dagger Q_{ij} + \mathbf{b}_{ij}^\dagger)u_j. \tag{3.8}$$

Inserting in (3.6) the expressions (3.3) for V_1, V_2, and using (3.4), (3.7), (3.8), one obtains

$$\begin{aligned} &\frac{\gamma}{2}x^\dagger M_1 x + \gamma \mathbf{n}_1^\dagger x + \gamma e_1 \\ &\quad = (x^\dagger M_1 + \mathbf{n}_1^\dagger) \cdot [Ax - B_1(B_1^\dagger(M_1 x + \mathbf{n}_1) + Q_{11}^\dagger x + \mathbf{b}_{11}) \\ &\qquad - B_2(B_2^\dagger(M_2 x + \mathbf{n}_2) + Q_{22}^\dagger x + \mathbf{b}_{22})] + \frac{1}{2}x^\dagger P_1 x + \mathbf{a}_1^\dagger x \\ &\qquad - \sum_{j=1,2}(x^\dagger Q_{1j} + \mathbf{b}_{1j}^\dagger)(B_j^\dagger(M_j x + \mathbf{n}_j) + Q_{jj}^\dagger x + \mathbf{b}_{jj}) \\ &\qquad + \frac{1}{2}((x^\dagger M_1 + \mathbf{n}_1^\dagger)B_1 + x^\dagger Q_{11} + \mathbf{b}_{11}^\dagger)(B_1^\dagger(M_1 x + \mathbf{n}_1) + Q_{11}^\dagger x + \mathbf{b}_{11}), \end{aligned} \tag{3.9}$$

and an entirely similar equation holds for V_2.

We regard (3.9) as an identity between two polynomials of degree 2 in the variables $x_1, \ldots, x_n$. Equating the coefficients of the quadratic terms, one obtains a system of algebraic equations for the coefficients of the symmetric matrices M_1, M_2. A direct computation yields

$$\begin{aligned} -\frac{1}{2}M_1 B_1 B_1^\dagger M_1 - M_1 B_2 B_2^\dagger M_2 + M_1 C - M_2 B_2 Q_{12}^\dagger + C_1 = 0, \\ -\frac{1}{2}M_2 B_2 B_2^\dagger M_2 - M_2 B_1 B_1^\dagger M_1 + M_2 C - M_1 B_1 Q_{21}^\dagger + C_2 = 0, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &C \doteq A - \frac{\gamma}{2}I - B_1 Q_{11}^\dagger - B_2 Q_{22}^\dagger, \\ &C_1 \doteq \frac{1}{2}P_1 - \frac{1}{2}Q_{11}Q_{11}^\dagger - Q_{12}Q_{22}^\dagger, \quad C_2 \doteq \frac{1}{2}P_2 - \frac{1}{2}Q_{22}Q_{22}^\dagger - Q_{21}Q_{11}^\dagger. \end{aligned} \tag{3.11}$$

Throughout the following, we assume that there exists a pair of $n \times n$ symmetric matrices M_1, M_2 providing a solution to the above system. Moreover, we assume that the resulting dynamics is asymptotically stable, so that all the eigenvalues of the matrix

$$Y \doteq A - (B_1 B_1^\dagger M_1 + B_2 B_2^\dagger M_2 + B_1 Q_{11}^\dagger + B_2 Q_{22}^\dagger) \tag{3.12}$$

have strictly negative real parts.

Next, equating the coefficients of the linear terms in (3.9), we obtain a system of linear equations for the vectors $\mathbf{n}_1, \mathbf{n}_2$:

$$\begin{pmatrix} (Y^\dagger - \gamma I) & -R_1 \\ -R_2 & (Y^\dagger - \gamma I) \end{pmatrix} \begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \tag{3.13}$$

where

$$R_1 \doteq M_2 B_2 B_2^\dagger + Q_{12} B_2^\dagger, \quad R_2 \doteq M_1 B_1 B_1^\dagger + Q_{21} B_1^\dagger,$$

$$T_1 = -\mathbf{a}_1 + \sum_{j=1,2} (M_1 B_j \mathbf{b}_{jj} + Q_{1j} \mathbf{b}_{jj} + Q_{jj} \mathbf{b}_{1j}) + M_2 B_2 \mathbf{b}_{12} - Q_{11} \mathbf{b}_{11}.$$

$$T_2 = -\mathbf{a}_2 + \sum_{j=1,2} (M_2 B_j \mathbf{b}_{jj} + Q_{2j} \mathbf{b}_{jj} + Q_{jj} \mathbf{b}_{2j}) + M_1 B_1 \mathbf{b}_{21} - Q_{22} \mathbf{b}_{22}.$$

Finally, equating the constant terms on the two sides of (3.9), one obtains an expression for e_1, e_2:

$$\gamma e_i = -\sum_{j=1,2} \mathbf{n}_i^\dagger B_j (B_j^\dagger \mathbf{n}_j + \mathbf{b}_{jj}) - \sum_{j=1,2} \mathbf{b}_{ij}^\dagger \mathbf{b}_{jj} + \frac{1}{2} \left| \mathbf{n}_1^\dagger B_1 + \mathbf{b}_{11}^\dagger \right|^2. \tag{3.14}$$

4. Affine Perturbations

Now consider the differential game (3.1)–(3.2). Let the feedback controls

$$u_i^*(x) = U_i x + \mathbf{v}_i, \quad i = 1, 2, \tag{4.1}$$

provide a Nash equilibrium solution. Let V_1, V_2 be the value functions for the two players, so that all the identities (3.3)–(3.14) hold.

We wish to understand whether this solution is stable with respect to iterations of the best reply map. In this section we study the case of affine perturbations, while the next section is concerned with general nonlinear perturbations. As we shall see, the answer is quite different in the two cases.

To state the question more precisely, consider two perturbed feedback controls

$$u_i^{(0)}(x) = U_i^{(0)} x + \mathbf{v}_i^{(0)}, \quad i = 1, 2. \tag{4.2}$$

By induction on N, define a sequence of feedback controls

$$u_i^{(k)}(x) = U_i^{(k)} x + \mathbf{v}_i^{(k)}, \quad i = 1, 2, \tag{4.3}$$

where $u_1^{(k)}$ is the optimal feedback for Player 1, in reply to the feedback $u_2^{(k-1)}$ implemented by Player 2, and similarly $u_2^{(k)}$ is the optimal feedback for Player 2, in reply to $u_1^{(k-1)}$. We seek conditions which guarantee that, if the pair $(u_1^{(0)}, u_2^{(0)})$ is sufficiently close to (u_1^*, u_2^*), then one has the convergence

$$(u_1^{(k)}, u_2^{(k)}) \;\to\; (u_1^*, u_2^*) \quad \text{as} \quad k \to \infty. \tag{4.4}$$

For this purpose, we consider the two maps

$$\Phi_1 : U_2^{(k-1)} \;\mapsto\; U_1^{(k)}, \quad \Phi_2 : U_1^{(k)} \;\mapsto\; U_2^{(k+1)}. \tag{4.5}$$

and compute the differential of the composite mapping $\Phi_2 \circ \Phi_1$ at the equilibrium point (U_1, U_2). Consider a small perturbation of the feedback for the first player:

$$u_{1,\varepsilon}(x) = (U_i + \varepsilon U_i')x + (\mathbf{v}_1 + \varepsilon \mathbf{v}_1').$$

Then, the optimal feedback for the second player will also be changed, say

$$u_{2,\varepsilon}(x) = (U_2 + \varepsilon U_2')x + \mathbf{v}_2 + \varepsilon \mathbf{v}_2' + o(\varepsilon).$$

The perturbation U_2' is computed as in (2.29), (2.27). Indeed, from the point of view of the second player, perturbing U_1 and $\mathbf{v}_1$ amounts to replacing the dynamics

$$\dot{x} = (A + B_1 U_1)x + B_2 u_2 + (B_1 \mathbf{v}_1 + \mathbf{f}) \tag{4.6}$$

with

$$\dot{x}_\varepsilon = (A + B_1(U_1 + \varepsilon U_1'))x + B_2 u_2 + (B_1(\mathbf{v}_1 + \varepsilon \mathbf{v}_1') + \mathbf{f}). \tag{4.7}$$

Moreover, the cost functional

$$J_2 = \int_0^\infty e^{-\gamma t}\left[\frac{1}{2}x^\dagger P_2 x + \mathbf{a}_2^\dagger x + \frac{|u_2|^2}{2} + (x^\dagger Q_{22} + \mathbf{b}_{22}^\dagger)u_2 \right.$$
$$\left. + (x^\dagger Q_{21} + \mathbf{b}_{21}^\dagger)(U_1 x + \mathbf{v}_1)\right] dt \tag{4.8}$$

is replaced by

$$\begin{aligned} J_{2,\varepsilon} = \int_0^\infty e^{-\gamma t}\Bigg[&\frac{1}{2}x^\dagger P_2 x + \mathbf{a}_2^\dagger x + \frac{|u_2|^2}{2} + (x^\dagger Q_{22} + \mathbf{b}_{22}^\dagger)u_2 \\ &+ (x^\dagger Q_{21} + \mathbf{b}_{21}^\dagger)(U_1 x + \varepsilon U_1' x + \mathbf{v}_1 + \varepsilon \mathbf{v}_1')\Bigg] dt. \end{aligned} \tag{4.9}$$

By the analysis in Sec. 2, the differential of the map Φ_2 in (4.5) is the linear map $U_1' \mapsto U_2'$ defined by

$$U_2' = -B_2^\dagger M_2', \tag{4.10}$$

$$M_2' = \int_0^\infty e^{-\gamma t}(e^{tY})^\dagger S_2 e^{tY}\, dt, \tag{4.11}$$

$$S_2 \doteq (M_2 A_2') + (M_2 A_2')^\dagger + P_2' = (M_2 B_1 U_1') + (M_2 B_1 U_1')^\dagger + (Q_{21} U_1') + (Q_{21} U_1')^\dagger. \tag{4.12}$$

The differential of the map Φ_1 is computed by the same formulas, (4.10)–(4.12), permuting the indices 1, 2.

We now consider the composition of two iterations, say $\widetilde{U}_1' = D\Phi_1(U_2') = D\Phi_1 \circ D\Phi_2(U_1')$. Observe that

$$\widetilde{U}_1' = B_1^\dagger \widetilde{M}_1, \quad U_2' = B_2^\dagger M_2', \quad U_1 = B_1^\dagger M_1',$$

where the matrices B_1, B_2 do not change from one iteration to the next. We have the estimate

$$\begin{aligned}
\|\widetilde{M_1'}\| &= \left\| \int_0^\infty e^{-\gamma t} (e^{tY})^\dagger \widetilde{S}_1 e^{tY}\, dt \right\| \\
&= \left\| \int_0^\infty e^{-\gamma t} (e^{tY})^\dagger ((M_1 B_2 U_2') + (M_1 B_2 U_2')^\dagger + (Q_{12} U_2') + (Q_{12} U_2')^\dagger) e^{tY}\, dt \right\| \\
&\le \left\| \int_0^\infty e^{-\gamma t} (e^{tY})^\dagger e^{tY}\, dt \right\| \cdot 2(\|M_1 B_2\| + \|Q_{12}\|)\|B_2\|\|M_2'\| \\
&\le \left\| \int_0^\infty e^{-\gamma t} (e^{tY})^\dagger e^{tY}\, dt \right\|^2 \cdot 4(\|M_1 B_2\| + \|Q_{12}\|)(\|M_2 B_1\| \\
&\quad + \|Q_{21}\|)\|B_2\|\|B_1\|\|M_1'\|.
\end{aligned} \tag{4.13}$$

Calling

$$\mathbf{I} \doteq \left\| \int_0^\infty e^{-\gamma t} (e^{tY})^\dagger e^{tY}\, dt \right\|,$$

we can now state

Theorem 1. *Let V_i, $i = 1, 2$ be the value functions corresponding to a Nash equilibrium feedback solution. Assume that*

$$4(\|M_1 B_2\| + \|Q_{12}\|)(\|M_2 B_1\| + \|Q_{21}\|)\|B_1\|\|B_2\| \cdot \mathbf{I}^2 < 1. \tag{4.14}$$

$$\left\| (\gamma I - Y)^{-1} \right\|^2 \|M_1 + Q_{12}\|\|M_2 + Q_{21}\| \cdot \|B_1\|\|B_2\| < 1. \tag{4.15}$$

Then, the Nash equilibrium is asymptotically stable with respect to iterations of the best reply map, within the class of piecewise affine feedback controls.

Proof. Consider the composite mapping

$$(U_1, \mathbf{v}_1) \ \mapsto \ (U_2, \mathbf{v}_2) \ \mapsto \ \widetilde{U}_1, \tilde{\mathbf{v}}_1), \tag{4.16}$$

where $u_2(x) = U_2 x + \mathbf{v}_2$ is the best reply for Player 2 to the feedback $u_1(x) = U_1 x + \mathbf{v}_1$ used by Player 1, while $\tilde{u}_1(x) = \widetilde{U}_1 x + \tilde{\mathbf{v}}_1$ is the best reply for Player 1 to the feedback $u_2(x) = U_2 x + \mathbf{v}_2$. Consider the corresponding value functions

$$\begin{aligned} V_1(x) &= \frac{1}{2}x^\dagger M_1 x + \mathbf{n}_1^\dagger x + e_1, \quad V_2(x) = \frac{1}{2}x^\dagger M_2 x + \mathbf{n}_2^\dagger x + e_2, \\ \widetilde{V}_1(x) &= \frac{1}{2}x^\dagger \widetilde{M}_1 x + \tilde{\mathbf{n}}_1^\dagger x + \tilde{e}_1. \end{aligned} \tag{4.17}$$

By (3.5)

$$U_i = -B_i^\dagger M_i + Q_{ii}^\dagger, \quad \mathbf{v}_i = -B_i^\dagger \mathbf{n}_i + \mathbf{b}_{ii}.$$

Therefore, to prove convergence of the iterates of the best reply map, it suffices to prove the convergence of the iterates of the composite map

$$(M_1, \mathbf{n}_1) \;\mapsto\; (M_2, \mathbf{n}_2) \;\mapsto\; (\widetilde{M}_1, \tilde{\mathbf{n}}_1). \tag{4.18}$$

Consider the differential of this mapping, computed at the Nash equilibrium:

$$\Lambda \doteq \begin{pmatrix} \partial\widetilde{M}_1/\partial M_1 & \partial\widetilde{M}_1/\partial \mathbf{n}_1 \\ \partial\tilde{\mathbf{n}}_1/\partial M_1 & \partial\tilde{\mathbf{n}}_1/\partial \mathbf{n}_1 \end{pmatrix}. \tag{4.19}$$

To prove the theorem, it suffices to show that the powers of this matrix converge to zero:

$$\Lambda^N \to 0 \quad \text{as} \quad N \to \infty. \tag{4.20}$$

Since $\partial\widetilde{M}_1/\partial\mathbf{n}_1 = 0$, the convergence in (4.20) will follow from the two inequalities

$$\|\partial\widetilde{M}_1/\partial M_1\| < 1, \quad \|\partial\tilde{\mathbf{n}}_1/\partial\mathbf{n}_1\| < 1. \tag{4.21}$$

By (4.13), the first inequality follows from the assumption (4.14).

Next, applying (2.30) with

$$\mathbf{c}' = \mathbf{v}_1' = -B_1^\dagger \mathbf{n}_1', \quad \mathbf{a}' = Q_{21}\mathbf{v}_1' = -Q_{21}B_1^\dagger \mathbf{n}_1',$$

we obtain

$$\mathbf{n}_2' = -(\gamma I - Y)^{-1}(M_2 B_1^\dagger + Q_{21}B_1^\dagger)\mathbf{n}_1'.$$

Similarly,

$$\tilde{\mathbf{n}}_1' = -(\gamma I - Y)^{-1}(M_1 B_2^\dagger + Q_{12}B_2^\dagger)\mathbf{n}_2'.$$

Hence,

$$\|\partial\tilde{\mathbf{n}}_1/\partial\mathbf{n}_1\| \le \left\|(\gamma I - Y)^{-1}\right\|^2 \|M_2 + Q_{21}\|\|M_1 + Q_{12}\| \cdot \|B_1\|\|B_2\|,$$

and the second inequality in (4.21) follows from the assumption (4.15). □

Example 1. The previous result applies, in particular, to a differential game with one weak player, considered in [2, 3]. Let the cost functionals be as in (3.2), but assume that the dynamics has the form

$$\dot{x} = Ax + B_1 u_1 + \theta B_2 u_2 + \mathbf{f}. \tag{4.22}$$

When $\theta = 0$, the Nash equilibrium solution is attained after one iteration. Indeed, in this case the second player cannot affect the evolution of the system. His best strategy is thus the myopic one:

$$u_2^*(x) = \operatorname*{argmin}_{\omega} \left\{ \frac{|\omega|^2}{2} + x^\dagger Q_{22}\omega + \mathbf{b}_{22}^\dagger \omega \right\} = -(Q_{22}^\dagger x + \mathbf{b}_{22}),$$

regardless of the feedback $u_1(\cdot)$ implemented by the first player.

On the other hand, for $\theta > 0$, replacing the term $\|B_2\|$ by $\|\theta B_2\|$ it is clear that the assumptions (4.14)–(4.15) are satisfied, as long as θ remains small enough.

5. Nonlinear Perturbations

The convergence result proved in the previous section strongly relied on the linear-quadratic structure of the game, and on the fact that all perturbed feedback controls remained within the class of affine functions of the state x.

In this section we show that, even for a linear-quadratic game, the iterates of the best reply map may fail to converge, as soon as we consider feedback controls which are not affine functions of the state.

Example 2. Consider the game with linear dynamics

$$\dot{x} = f(x, u_1, u_2) \doteq -x + u_1 + u_2 \tag{5.1}$$

and quadratic payoff functionals

$$J_1 \doteq \int_0^\infty e^{-t} \left[ax - \frac{u_1^2}{2} \right] dt, \tag{5.2}$$

$$J_2 \doteq \int_0^\infty e^{-t} \left[bx - \frac{u_2^2}{2} \right] dt. \tag{5.3}$$

We assume here $0 < a < b$. Call $V_1(x)$, $V_2(x)$ the value functions for the two players. Their derivatives will be written as

$$\xi \doteq V_1' = V_{1,x}, \quad \eta \doteq V_2' = V_{2,x}.$$

The optimal controls are then computed as

$$u_1^*(x,\xi) = \underset{\omega}{\operatorname{argmax}}\left(\xi\cdot\omega - \frac{\omega^2}{2}\right) = \xi, \tag{5.4}$$

$$u_2^*(x,\eta) = \underset{\omega}{\operatorname{argmax}}\left(\eta\cdot\omega - \frac{\omega^2}{2}\right) = \eta. \tag{5.5}$$

The value functions V_1, V_2 can be found by solving the system of Hamilton–Jacobi equations

$$\begin{cases} V_1 = H(x, V_1', V_2'), \\ V_2 = K(x, V_1', V_2'), \end{cases} \tag{5.6}$$

where

$$\begin{aligned} H(x,\xi,\eta) &= \xi\cdot(-x + u^*(x,\xi) + v^*(x,\eta)) + \left(ax - \frac{(u^*(x,\xi))^2}{2}\right) \\ &= \frac{\xi^2}{2} + (a-\xi)x + \xi\eta, \end{aligned} \tag{5.7}$$

$$\begin{aligned} K(x,\xi,\eta) &= \eta\cdot(-x + u^*(x,\xi) + v^*(x,\eta)) + \left(bx - \frac{(v^*(x,\eta))^2}{2}\right) \\ &= \xi\eta + (b-\eta)x + \frac{\eta^2}{2}. \end{aligned} \tag{5.8}$$

Notice that

$$H_\xi = K_\eta = -x + \xi + \eta = f(x,\xi,\eta). \tag{5.9}$$

Differentiating (5.6), we obtain the system

$$\begin{cases} \xi = H_x + H_\xi \xi' + H_\eta \eta', \\ \eta = K_x + K_\xi \xi' + K_\eta \eta'. \end{cases} \tag{5.10}$$

In matrix notation, this can be written as

$$\begin{pmatrix} \xi+\eta-x & \xi \\ \eta & \xi+\eta-x \end{pmatrix}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} 2\xi - a \\ 2\eta - b \end{pmatrix}. \tag{5.11}$$

The system (5.11) admits the constant solution

$$\xi \equiv \frac{a}{2}, \quad \eta \equiv \frac{b}{2}. \tag{5.12}$$

This correspond to a Nash equilibrium solution of the differential game consisting of two constant feedback controls:

$$u_1^*(x) = \frac{a}{2}, \quad u_2^*(x) = \frac{b}{2}. \tag{5.13}$$

Next, we study whether there exist other solutions, possibly described by nonlinear feedbacks $u_1(x)$, $u_2(x)$. Following [2, 3], we write (5.11) as a Pfaffian system

$$\begin{cases} \omega_1 \doteq (a-2\xi)\,dx + (\xi+\eta-x)\,d\xi + \xi\,d\eta = 0, \\ \omega_2 \doteq (b-2\eta)\,dx + \eta\,d\xi + (\xi+\eta-x)\,d\eta = 0. \end{cases} \tag{5.14}$$

The graph of a solution to (5.11) can then be obtained by suitably concatenating trajectories of the vector field

$$\mathbf{v} = \omega_1 \wedge \omega_2 = \begin{pmatrix} (\xi+\eta-x)^2 - \xi\eta \\ (\xi+\eta-x)(2\xi-a) - \xi(2\eta-b) \\ (\xi+\eta-x)(2\eta-b) - \eta(2\xi-a) \end{pmatrix}. \tag{5.15}$$

Notice that the first component of $\mathbf{v}$ vanishes along the conical surface

$$\Gamma \doteq \{(x,\xi,\eta);\ (\xi+\eta-x)^2 - \xi\eta = 0\}. \tag{5.16}$$

At a point $P \in \Gamma$ we either have $\mathbf{v}(P) = 0 \in \mathbb{R}^3$, or else the vector $\mathbf{v}(P)$ is vertical. The only way to connect trajectories of the vector field $\mathbf{v}$ forming the graph of a smooth function $x \mapsto W(x) = (y(x),\, z(x))$ is to cross the surface Γ somewhere along the two curves where $\mathbf{v}$ vanishes, namely

$$\gamma^\pm \doteq \left\{(x,\xi,\eta);\ x = (\xi+\eta) \pm \sqrt{\xi\eta},\ (2\xi-a) = \pm\sqrt{\frac{\xi}{\eta}}(2\eta-b)\right\}. \tag{5.17}$$

Observe that the map

$$t \mapsto P(t) \doteq \left(x(t), \frac{a}{2}, \frac{b}{2}\right), \quad x(t) = \frac{p^+ + p^- e^{\kappa t}}{1+e^{\kappa t}}, \quad \kappa = \frac{1}{\sqrt{ab}}, \quad p^\pm = \frac{a+b\pm\sqrt{ab}}{2} \tag{5.18}$$

describes a heteroclinic orbit of the vector field $\mathbf{v}$ in (5.15), connecting the two stationary points

$$P^+ = \left(\frac{a+b+\sqrt{ab}}{2}, \frac{a}{2}, \frac{b}{2}\right) \in \gamma^+, \quad P^- = \left(\frac{a+b-\sqrt{ab}}{2}, \frac{a}{2}, \frac{b}{2}\right) \in \gamma^-.$$

Setting $X(t) \doteq a+b-2x(t)$, the Jacobian matrix of the vector field $\mathbf{v}$ at the point $P(t)$ is

$$A(t) = \begin{pmatrix} -X(t) & X(t)-\frac{b}{2} & X(t)-\frac{a}{2} \\ 0 & X(t) & -a \\ 0 & -b & X(t) \end{pmatrix}. \tag{5.19}$$

The eigenvalues λ_i and the corresponding eigenvectors $\mathbf{w}_i$ of the matrix $A(t)$ are:

$$\lambda_1(t) = -X(t), \quad \lambda_2(t) = X(t) + \sqrt{ab}, \quad \lambda_3(t) = X(t) - \sqrt{ab}, \tag{5.20}$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 - \sqrt{\frac{b}{a}} \\ 2 \\ -2\sqrt{\frac{b}{a}} \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 1 + \sqrt{\frac{b}{a}} \\ 2 \\ 2\sqrt{\frac{b}{a}} \end{pmatrix}. \tag{5.21}$$

In particular, at the point P^- the three eigenvalues are $-\sqrt{ab} < 0 < 2\sqrt{ab}$, while at the point P^+ the eigenvalues are $-2\sqrt{ab} < 0 < \sqrt{ab}$. Since the eigenvectors are constant, the general solution to the linear equation

$$\dot{y} = A(t)y$$

can be written as

$$y(t) = c_1 \exp\left\{\int_0^t \lambda_1(s)ds\right\} \mathbf{w}_1 + c_2 \exp\left\{\int_0^t \lambda_2(s)ds\right\} \mathbf{w}_2$$
$$+ c_3 \exp\left\{\int_0^t \lambda_3(s)ds\right\} \mathbf{w}_3, \tag{5.22}$$

for some constants c_1, c_2, c_3. By (5.18)–(5.20), we have

$$\begin{aligned} &\lim_{t\to-\infty} |y(t)| = +\infty \quad \text{if and only if} \quad c_3 \neq 0, \\ &\lim_{t\to+\infty} |y(t)| = +\infty \quad \text{if and only if} \quad c_2 \neq 0. \end{aligned} \tag{5.23}$$

Let us call Σ^+ the 2-dimensional manifold of points $\overline{P} = (\bar{x}, \bar{\xi}, \bar{\eta})$ such that the solution to

$$\dot{P} = \mathbf{v}(P), \quad P(0) = \overline{P} \tag{5.24}$$

satisfies $P(t) \to \gamma^+$ as $t \to -\infty$. Similarly, call Σ^- the 2-dimensional manifold of points $\overline{P} = (\bar{x}, \bar{\xi}, \bar{\eta})$ such that the solution of (5.24) satisfies $P(t) \to \gamma^-$ as $t \to +\infty$. By (5.23), these two manifolds intersect transversally along the segment P^-P^+. We conclude that there can be no other solutions to (5.11), in a neighborhood of the constant solution (5.12).

We claim that this pair of feedback controls, providing a fixed point of the "best reply" map is unstable with respect to nonlinear perturbations. More precisely:

Proposition 1. *There exists $\delta > 0$ and a sequence of smooth perturbations $\phi_k \in \mathcal{C}_c^\infty(\mathbb{R})$ such that the following holds.*

- *The $\mathcal{C}^k$ norms of ϕ_k satisfy*

$$\|\phi_k\|_{\mathcal{C}^k} \to 0. \tag{5.25}$$

- *For any $k \geq 1$, starting with the values*

$$u_1^{(0)}(x) = \frac{a}{2} + \phi_k(x), \quad u_2^{(0)}(x) = \frac{b}{2}, \tag{5.26}$$

the iterates $(u_1^{(N)}, u_2^{(N)})$ of the best reply map do not converge to the solution (u_1^*, u_2^*) in (5.13). *Indeed,*

$$\limsup_{N\to\infty} \left\{ \left|u_1^{(N)}(x) - \frac{a}{2}\right| + \left|u_2^{(N)}(x) - \frac{b}{2}\right|;\ \frac{a+b}{2} < x < \frac{a+b+6\delta}{2} \right\} > \delta. \tag{5.27}$$

Proof. To construct the nonlinear perturbations ϕ_k, choose $0 < \delta \ll 1$ and let $x_0 \doteq \frac{a+b}{2} + 3\delta$. Consider the standard $\mathcal{C}^\infty$ function with compact support

$$\phi(s) = \begin{cases} \exp\left\{\dfrac{1}{1-s^2}\right\} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then, define

$$\phi_k(x) \doteq c_k\, \phi\left(\frac{x - x_0}{\varepsilon_k}\right), \quad k \geq 1.$$

Choosing sequences of numbers numbers $0 < c_k \ll \varepsilon_k$ converging to zero sufficiently fast, the condition (5.25) can be easily satisfied.

We now examine the sequence of iterations of the best reply map, starting from the initial feedbacks

$$\xi_{(0)}(x) = \frac{a}{2} + \psi_k(x), \quad \eta_{(0)}(x) = \frac{b}{2}. \tag{5.28}$$

Assume $\xi_{(N)}, \eta_{(N)}$ have been determined. By (5.11), the next iteration yields a pair of functions $\xi_{(N+1)}(x)$ and $\eta_{(N+1)}(x)$, respectively providing solutions to

$$\begin{cases} \xi' = \dfrac{1}{\xi + \eta_{(N)}(x) - x} \cdot (2\xi - a - \eta'_{(N)}(x)\xi), \\ \eta' = \dfrac{1}{\eta + \xi_{(N)}(x) - x} \cdot (2\eta - b - \xi'_{(N)}(x)\,\eta). \end{cases} \tag{5.29}$$

Iterating once more, we find that $\xi_{(N+2)}$ provides a solution to

$$\xi' = \frac{1}{\xi + \eta_{(N+1)}(x) - x} \cdot \left(2\xi - a - \left[\frac{1}{\eta_{(N+1)} + \xi_{(N)}(x) - x} \cdot (2\eta_{(N+1)} - b - \xi'_{(N)}(x)\,\eta)\right] \xi\right). \tag{5.30}$$

At this stage, one should observe that the system of ODEs (5.11), as well as the two ODEs in (5.29), are not supplemented by initial data. It is the singularity in the coefficients that determines one particular, globally defined solution. For the system (5.11), the 2×2 matrix of coefficients fails to be invertible on the conical surface Γ

at (5.16). A globally smooth solution is found by concatenating trajectories of the vector field $\mathbf{v}$ in (5.15). As we have seen, these trajectories must cross Γ at points where $\mathbf{v}$ vanishes. Since the manifolds Σ^+, Σ^- intersect transversally, the segment P^+P^- is the only heteroclinic orbit, and the unique Nash equilibrium solution is (5.13).

On the other hand, the right-hand sides of (5.29) are both singular at points where $\xi+\eta-x=0$. At each iteration, the only solution which is regular across the surface where $\xi+\eta-x=0$ is the one for which

$$\xi_{(N)}(x) = \frac{a}{2}, \quad \eta_{(N)}(x) = \frac{b}{2} \quad \text{for all } x \leq x_0 - \varepsilon_k, \quad N \geq 0. \tag{5.31}$$

The functions $\xi_{(N+1)}, \eta_{(N+1)}$ can thus be constructed by solving (5.29) on the half line $[x_0 - \varepsilon_k\,, +\infty[$, with boundary data

$$\xi(x_0 - \varepsilon_k) = 0, \quad \eta(x_0 - \varepsilon_k) = 0. \tag{5.32}$$

Observe that the Cauchy problem for the system

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \frac{1}{(\xi+\eta-x)^2 - \xi\eta} \begin{pmatrix} \xi+\eta-x & -\xi \\ -\eta & \xi+\eta-x \end{pmatrix} \begin{pmatrix} 2\xi - a \\ 2\eta - b \end{pmatrix}$$

with initial data (5.32) is well posed in a neighborhood of x_0, and its solution provides a fixed point of the iteration scheme (5.29). However, this iteration scheme, motivated by the best-reply map, is highly unstable for x close to $\frac{a+b}{2}$. Indeed, contrary to the usual Picard integral map, the right-hand sides of (5.29) depend also on the derivatives $\xi'_{(N)},\ \eta'_{(N)}$. Since $\xi \approx a/2$, $\eta \approx b/2$, $x \approx (a+b)/2 + 3\delta$, from (5.30) it follows

$$\xi'_{(N+2)}(x) \approx \left(\frac{1}{\frac{a}{2} + \frac{b}{2} - x} \right)^2 \frac{ab}{4} \xi'_{(N)}(x).$$

We thus expect that $\xi'_{(N)}(x) \to +\infty$ for $x_0 - \varepsilon_k < x < x_0$. A more careful argument is as follows. Assume

$$\frac{a}{2} \leq \xi_{(N)}(x) \leq \frac{a}{2} + \delta, \quad \frac{b}{2} - \delta \leq \eta_{(N+1)}(x) \leq \frac{b}{2} + \delta, \quad \text{for all } x \ \in [x_0 - \varepsilon_k, \bar{y}], \tag{5.33}$$

for some $\bar{y} \in [x_0 - \varepsilon_k, x_0]$. Then, (5.30) yields

$$\xi'_{(N+2)}(x) \geq \frac{1}{(3\delta)^2} \frac{ab}{4}\, \xi'_{(N)}(x) - \frac{1}{\delta^2}(2\eta_{(N+1)}(x) - b). \tag{5.34}$$

Moreover, the second equation in (5.29) yields

$$\eta'_{(N+1)}(x) = \frac{1}{x - \eta - \xi_{(N)}(x)}[\xi'_{(N)}(x) \cdot \eta - 2\eta + b] \le \frac{1}{\delta}\left(\frac{b}{2} + 2\delta\right) \xi'_{(N)}(x). \tag{5.35}$$

Hence, integrating and inserting in (5.34), we obtain

$$\xi'_{(N+2)}(x) \ge \frac{1}{(3\delta)^2}\frac{ab}{4}\xi'_{(N)}(x) - \frac{1}{\delta^2}\left(\frac{b}{\delta} + 4\right) \int_{x_0-\varepsilon_k}^{x} \xi'_{(N)}(y)\, dy. \tag{5.36}$$

It is not restrictive to assume that $\varepsilon_k > 0$ is so small that $(\frac{b}{\delta} + 4)\varepsilon_k \le \frac{ab}{72}$. If this holds, for all $x \in [x_0 - \varepsilon_k\,, \bar{y}]$, one has

$$\begin{aligned} \xi_{(N+2)}(x) - \frac{a}{2} &= \int_{x_0-\varepsilon_k}^{x} \xi'_{(N+2)}(y)\, dy \\ &\ge \frac{1}{(3\delta)^2}\frac{ab}{4} \cdot \int_{x_0-\varepsilon_k}^{x} \xi'_{(N)}(y)\, dy \\ &\quad - \frac{1}{\delta^2}\left(\frac{b}{\delta} + 4\right) \cdot [x - (x_0 - \varepsilon_k)] \int_{x_0-\varepsilon_k}^{x} \xi'_{(N)}(y)\, dy \\ &\ge \frac{1}{(3\delta)^2}\frac{ab}{8} \cdot \int_{x_0-\varepsilon_k}^{x} \xi'_{(N)}(y)\, dy = \frac{1}{(3\delta)^2}\frac{ab}{8} \cdot \left(\xi_{(N)}(x) - \frac{a}{2}\right). \end{aligned} \tag{5.37}$$

Since $0 < \delta \ll 1$, the above estimate shows that, for every $x_0 - \varepsilon_k < x < \bar{y}$, the sequence $\xi_{(N)}(x) - a/2$ keeps increasing, as long as the bounds (5.33) hold. By continuity, for any given $N^* \ge 1$ we can now find $y(N^*) > x_0 - \varepsilon_k$ such that the estimates (5.33) hold for all $N \le N^*$ and $x \in [x_0 - \varepsilon_k,\, y(N^*)]$. By (5.37), the sequence $\xi_{(N)}(x)$ keeps increasing. Hence there exists $N > N^*$ such that one of the bounds in (5.33) fails at some $x \in [x_0 - \varepsilon_k,\, y(N^*)]$. This establishes the limit (5.27). □

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